Un auto-morphisme bidendriforme de WQSym

Journée de Combinatoire: IRMA

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Exemples d'algèbres de Hopf

- Arbres binaires, **PBT**, Loday-Ronco
- Fonctions symétriques non-commutatives, **Sym**
- Fonctions quasi-symétriques, *QSym*
- Permutations, **FQSym**, Malvenuto-Reutenauer
- Mots tassés, **WQSym**, Hivert

Mots tassés

Définition

Un mot sur l'alphabet $\mathbb{N}_{>0}$ est dit **tassé** si toutes les lettres de 1 à son maximum m apparaissent au moins une fois.

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Mots tassés de tailles 0, 1, 2 et 3

- ϵ

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- ϵ
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- 12 21 11

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- ϵ
- 1
- 12 21 11
- 123 132 213 231 312 321
  122 212 221 112 121 211 111

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Mots tassés de tailles 0, 1, 2 et 3

- ϵ
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- 12 21 11
- 123 132 213 231 312 321
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Mots tassés de taille n [OEIS A000670]

n	1	2	3	4	5	6	7	8
$\mathbf{PW}_n$	1	3	13	75	541	4683	47293	545835

Tassement

Exemple

$24154 \notin \mathbf{PW}$

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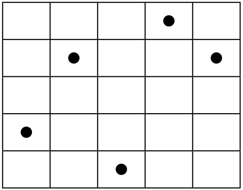
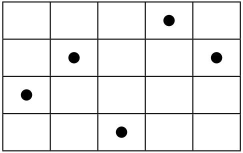
Une représentation : $\#\text{lignes} \leq \#\text{colonnes}$



2 4 1 5 4

retrait lignes vides

$\rightarrow$ pack $\rightarrow$



2 3 1 4 3

Algèbre de Hopf

Exemple

WQSym

- $3112 + 212 - 3\ 212341 - \frac{5}{3}\ 111$

Algèbre de Hopf

Exemple

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$

Algèbre de Hopf

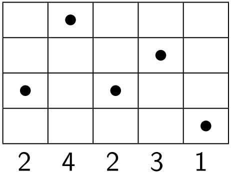
Exemple

WQSym

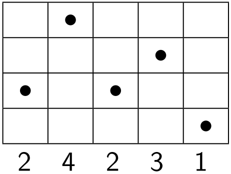
- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
- $\mathbb{R}_{12}\mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\Delta(\mathbb{R}_{24231}) = \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_{1} + \mathbb{R}_{24231} \otimes \mathbb{R}_{\epsilon}$

- Un produit associatif unitaire $\cdot$
- Un coproduit coassociatif counitaire Δ
- La relation de Hopf $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

Déconcaténation réduite

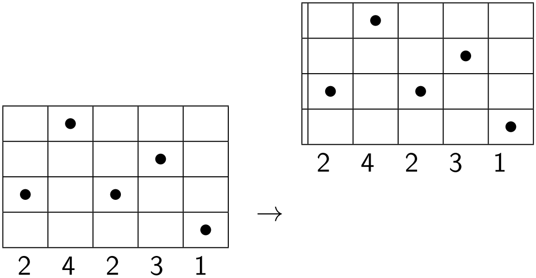


Déconcaténation réduite

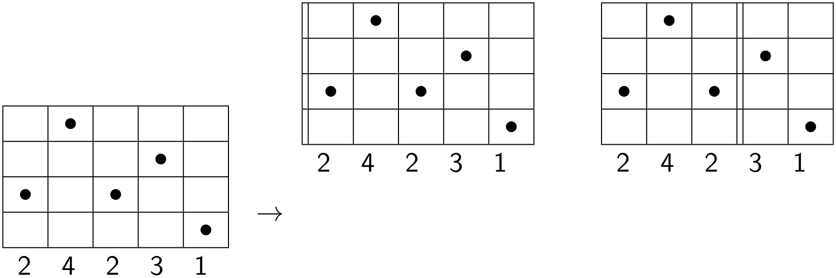


$\rightarrow$

Déconcaténation réduite



Déconcaténation réduite



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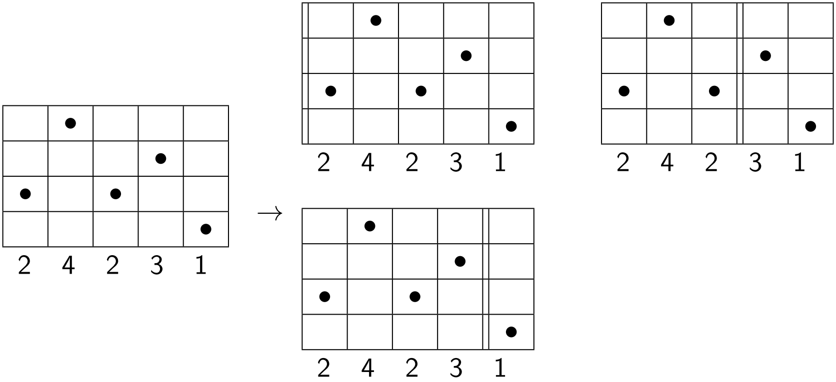
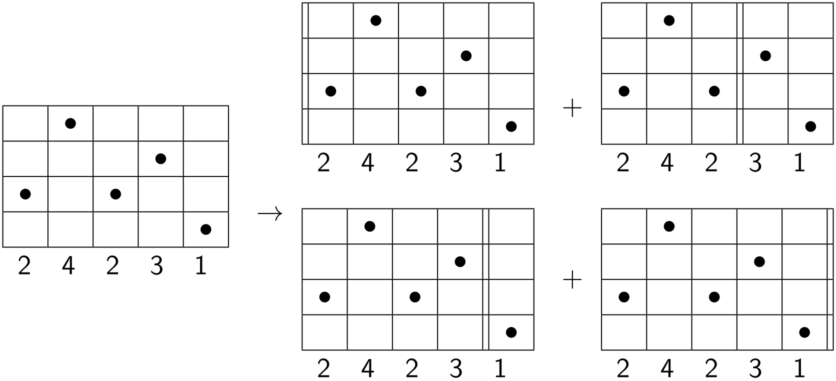
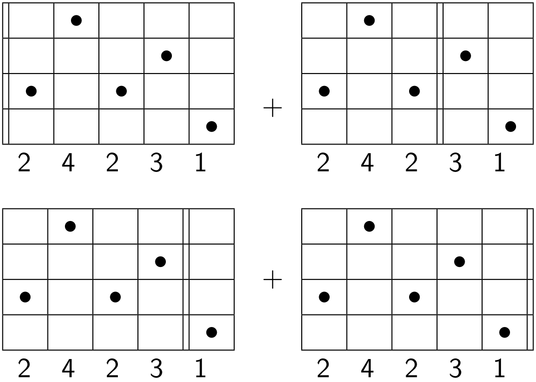


Déconcaténation réduite



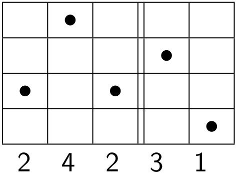
Déconcaténation réduite

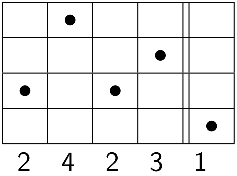
$\mathbb{R}_{24231} \to$



Déconcaténation réduite

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} + $$





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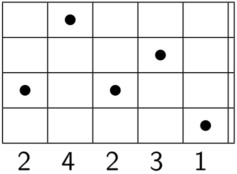

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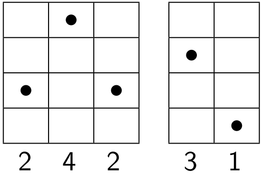
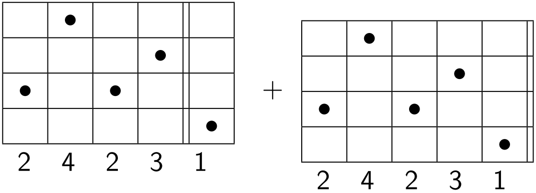
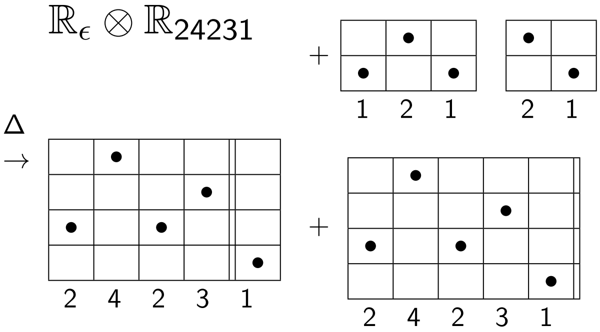




Déconcaténation réduite

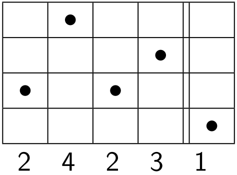
$\mathbb{R}_{24231}$



Déconcaténation réduite

$$\mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21}$$

$$\mathbb{R}_{24231} \xrightarrow{\Delta}$$



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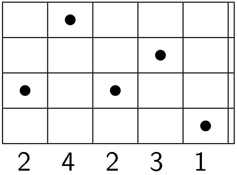
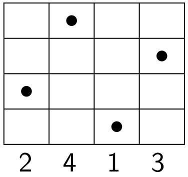


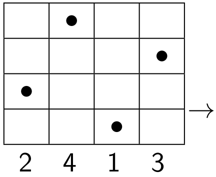
Déconcaténation réduite

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \begin{array}{ccc} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} & + & \mathbb{R}_{121} \otimes \mathbb{R}_{21} \\ \mathbb{R}_{1312} \otimes \mathbb{R}_{1} & + & \mathbb{R}_{24231} \otimes \mathbb{R}_{\epsilon} \end{array}$$

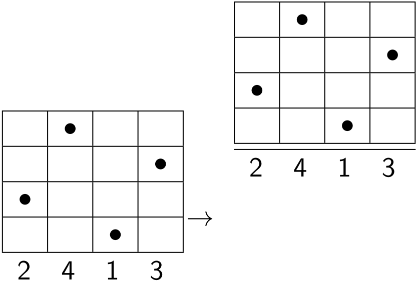
Désassemblage horizontal



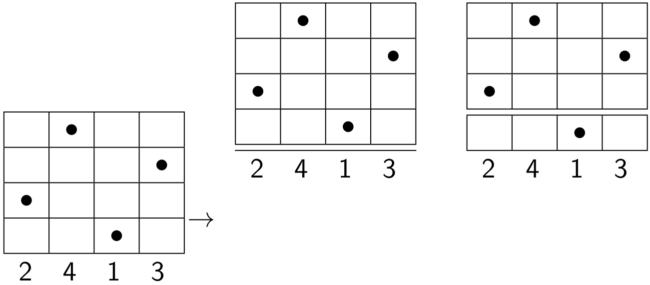
Désassemblage horizontal



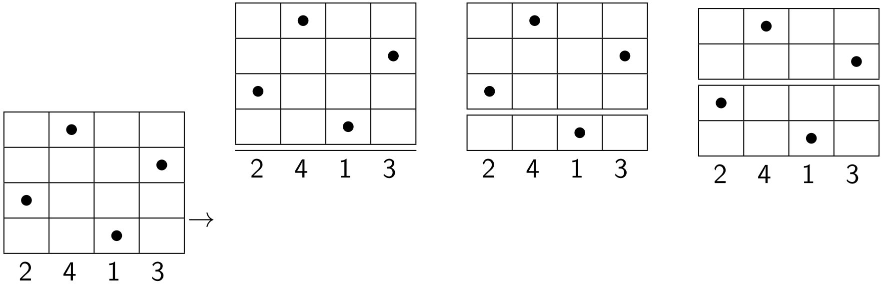
Désassemblage horizontal



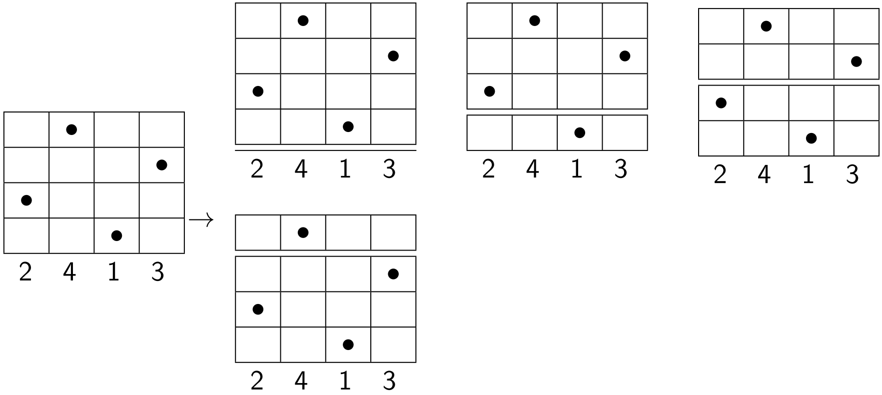
Désassemblage horizontal



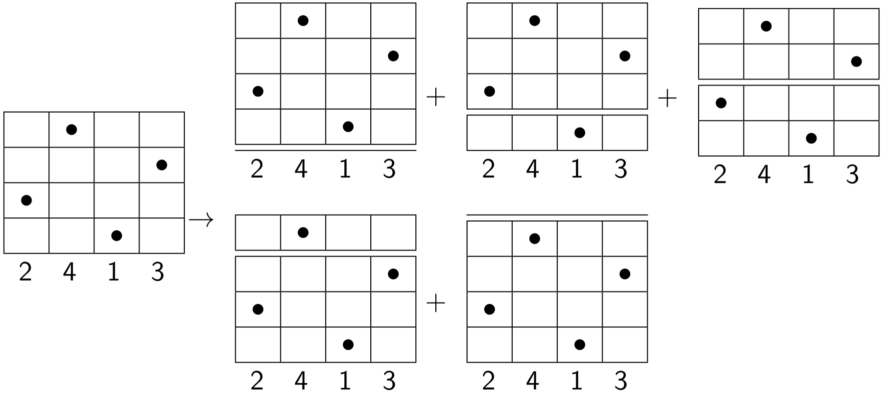
Désassemblage horizontal



Désassemblage horizontal

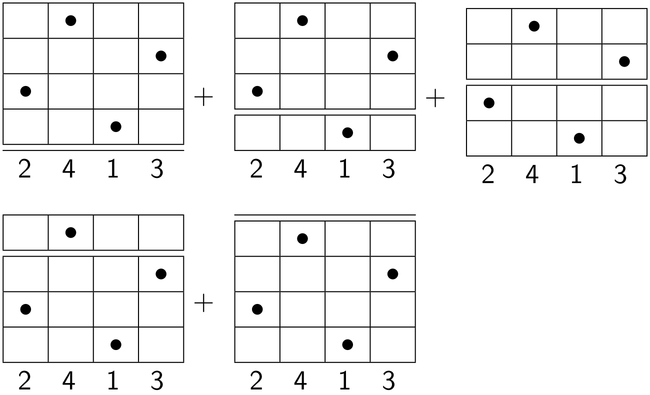


Désassemblage horizontal



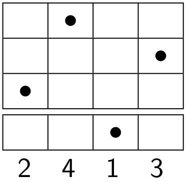
Désassemblage horizontal

$$\mathbb{Q}_{2413} \xrightarrow{\Delta}$$



Désassemblage horizontal

$$\mathbb{Q}_{2413} \xrightarrow{\Delta} \mathbb{Q}_\epsilon \otimes \mathbb{Q}_{2413} +$$



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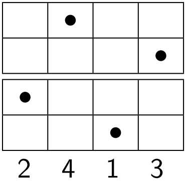

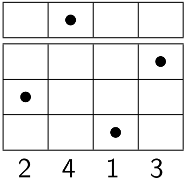

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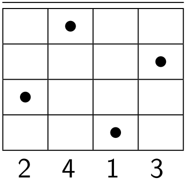
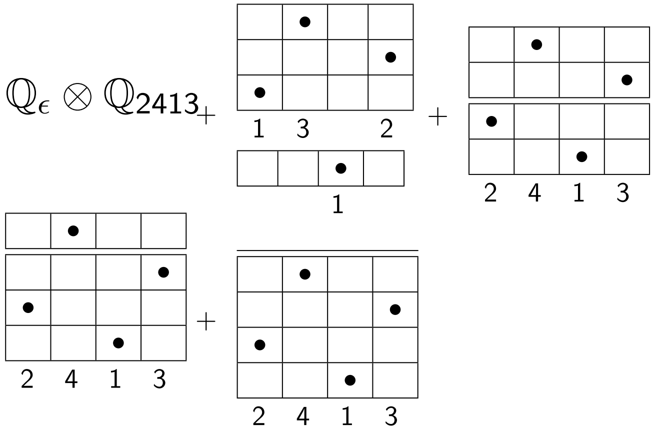


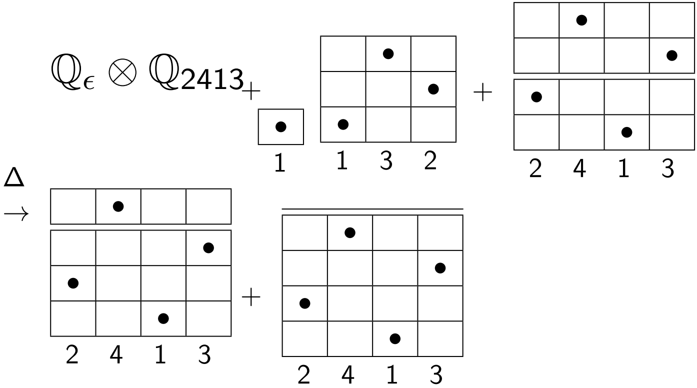
Désassemblage horizontal

$\mathbb{Q}_{2413} \xrightarrow{\Delta}$

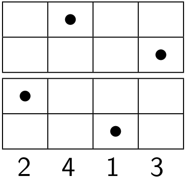


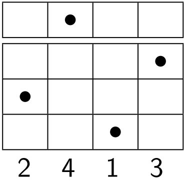
Désassemblage horizontal

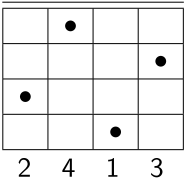
$\mathbb{Q}_{2413}$



Désassemblage horizontal

$\mathbb{Q}_{\epsilon} \otimes \mathbb{Q}_{2413} + \mathbb{Q}_{1} \otimes \mathbb{Q}_{132} +$ 

$\mathbb{Q}_{2413} \xrightarrow{\Delta}$ 

$+$ 

Désassemblage horizontal

$$\mathbb{Q}_{2413} \xrightarrow{\Delta} \mathbb{Q}_{\epsilon} \otimes \mathbb{Q}_{2413} + \mathbb{Q}_{1} \otimes \mathbb{Q}_{132} + \mathbb{Q}_{21} \otimes \mathbb{Q}_{21} + \mathbb{Q}_{213} \otimes \mathbb{Q}_{1} + \mathbb{Q}_{2413} \otimes \mathbb{Q}_{\epsilon}$$

Auto-dualité

- $\mathbb{R}$ et $\mathbb{Q}$ bases de **WQSym** et **WQSym***

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- 2001 Duchanp-Hivert-Thibon conjecturent l'auto-dualité de **WQSym**
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- Pas d'isomorphisme explicite

Demis coproduits

Exemple de coproduits gauche et droit

- $$\tilde{\Delta}(\mathbb{Q}_{12536434}) = \mathbb{Q}_1 \otimes \mathbb{Q}_{1425323} + \mathbb{Q}_{12} \otimes \mathbb{Q}_{314212} + \mathbb{Q}_{1233} \otimes \mathbb{Q}_{2311} + \mathbb{Q}_{123434} \otimes \mathbb{Q}_{12} + \mathbb{Q}_{1253434} \otimes \mathbb{Q}_1$$

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- $\Delta_{\succeq}(\mathbb{Q}_{12536434}) = \mathbb{Q}_1 \otimes \mathbb{Q}_{1425323} + \mathbb{Q}_{12} \otimes \mathbb{Q}_{314212} + \mathbb{Q}_{1233} \otimes \mathbb{Q}_{2311}$,
- $\Delta_{\preceq}(\mathbb{Q}_{12536434}) = \mathbb{Q}_{123434} \otimes \mathbb{Q}_{12} + \mathbb{Q}_{1253434} \otimes \mathbb{Q}_1$.

Demis coproduits

Définitions

- $\Delta_{\succeq}(\mathbb{Q}_u) := \sum_{i=1}^{u_n-1} \mathbb{Q}_{u|_{\leq i}} \otimes \mathbb{Q}_{\mathrm{pack}(u|_{>i})}$,
- $\Delta_{\preceq}(\mathbb{Q}_u) := \sum_{i=u_n}^{\max(u)-1} \mathbb{Q}_{u|_{\leq i}} \otimes \mathbb{Q}_{\mathrm{pack}(u|_{>i})}$.

Exemple de coproduits gauche et droit

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Bigèbre bidendriforme

Définition

- Raffinement de l'associativité et la coassociativité
  - 3 et 3 équations
- Raffinement de la relation de Hopf
  - 4 équations

Bigèbre bidendriforme

$$
\begin{aligned}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c),\\
(a \succ b) \prec c &= a \succ (b \prec c),\\
(a \prec b + a \succ b) \succ c &= a \succ (b \succ c).
\end{aligned}
$$

$$
\begin{aligned}
(\Delta_\prec \otimes \mathrm{Id}) \circ \Delta_\prec(a) &= (\mathrm{Id} \otimes \Delta_\prec + \mathrm{Id} \otimes \Delta_\succ) \circ \Delta_\prec(a),\\
(\Delta_\succ \otimes \mathrm{Id}) \circ \Delta_\prec(a) &= (\mathrm{Id} \otimes \Delta_\prec) \circ \Delta_\succ(a),\\
(\Delta_\prec \otimes \mathrm{Id} + \Delta_\succ \otimes \mathrm{Id}) \circ \Delta_\succ(a) &= (\mathrm{Id} \otimes \Delta_\succ) \circ \Delta_\succ(a).
\end{aligned}
$$

$$
\begin{aligned}
&\Delta_\succ(a \succ b) = a'b'_\succ \otimes a'' \succ b''_\succ + b'_\succ \otimes a \succ b''_\succ + ab'_\succ \otimes b''_\succ + a' \otimes a'' \succ b + a \otimes b,\\
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&\Delta_\prec(a \prec b) = a'b'_\prec \otimes a'' \prec b''_\prec + b'_\prec \otimes a \prec b''_\prec + a'b \otimes a'' + b \otimes a.
\end{aligned}
$$

Bigèbre bidendriforme

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Théorème [Foissy]

Si A est une bigèbre bidendriforme alors A est généré librement par $\mathrm{TPrim}(A)$ en tant qu'algèbre dendriforme.

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Séries

n	1	2	3	4	5	6	7	8
$\mathbf{WQSym}_n$	1	3	13	75	541	4 683	47 293	545 835
TPrim_n	1	1	4	28	240	2 384	26 832	337 168

Bigèbre bidendriforme

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Corollaire

WQSym est auto-duale.

Définitions

Élément primitif

P est un éléments primitif $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

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Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

Définitions

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Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

Élément totalement primitif

P est une élément totalement primitif $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

Définitions

Élément primitif

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Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

Élément totalement primitif

P est une élément totalement primitif $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

$\tilde{\Delta}(\mathbb{R}_{12443}) = \mathbb{R}_{1233} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{12} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{1332}$

$\tilde{\Delta}(\mathbb{R}_{21443}) = \mathbb{R}_{2133} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{21} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{1332}$

$\tilde{\Delta}(\mathbb{R}_{23441}) = \mathbb{R}_{1233} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{12} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{2331}$

$\tilde{\Delta}(\mathbb{R}_{32441}) = \mathbb{R}_{2133} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{21} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{2331}$

Mon but

Isomorphisme bidendriforme explicite entre **WQSym** et sa duale

Mon but

Isomorphisme bidendriforme explicite entre **WQSym** et sa duale

$$\Updownarrow$$

Isomorphisme explicite entre TPrim(**WQSym**) et le dual

Mon but

Isomorphisme bidendriforme explicite entre **WQSym** et sa duale

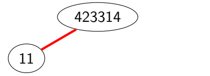
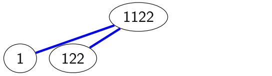
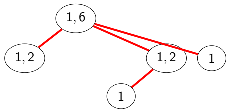
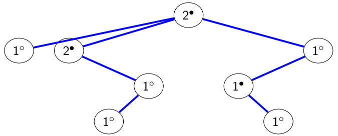
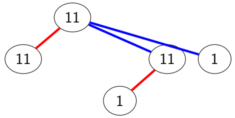
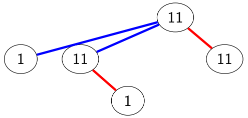
$$\Updownarrow$$

Isomorphisme explicite entre TPrim(**WQSym**) et le dual

Construction de deux bases de totalement primitif
(dans **WQSym** et **WQSym***)

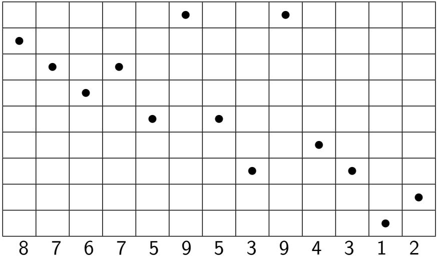
Forêts biplanes, représentation de décompositions

44523315

	Squelettes	
	Arbres tassés	
	Arbres bicolores	

Squelette rouge de 8767595394312

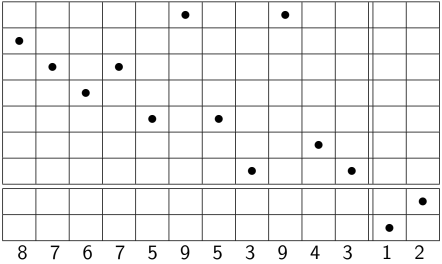
$F_{\color{red}{R}\mathrm{ske}}(8767595394312)$



Squelette rouge de 8767595394312

$F_{\color{red}{R}\mathrm{ske}}(8767595394312)$

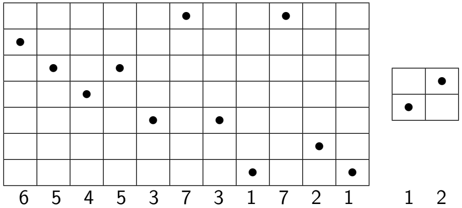
Factorisation en descentes globales



Squelette rouge de 8767595394312

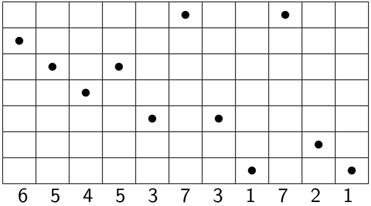
$F_{\color{red}{\mathrm{R}}\mathrm{ske}}(8767595394312) =$
$T_{\color{red}{\mathrm{R}}\mathrm{ske}}(65453731721)\, T_{\color{red}{\mathrm{R}}\mathrm{ske}}(12)$

Factorisation en descentes globales
$+$ tassement



Squelette rouge de 8767595394312

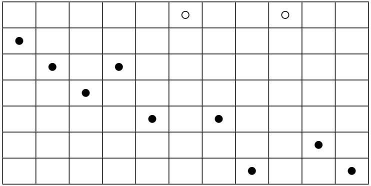
$$\mathrm{F}_{\color{red}{\mathrm{R}}\mathrm{ske}}(8767595394312) =$$
$$\mathrm{T}_{\color{red}{\mathrm{R}}\mathrm{ske}}(65453731721)\ \mathrm{T}_{\color{red}{\mathrm{R}}\mathrm{ske}}(12)$$



Squelette rouge de 8767595394312

$F_{\color{red}{R}\mathrm{ske}}(8767595394312) =$
$T_{\color{red}{R}\mathrm{ske}}(65453731721)\, T_{\color{red}{R}\mathrm{ske}}(12)$

Retrait des lettres de valeur max

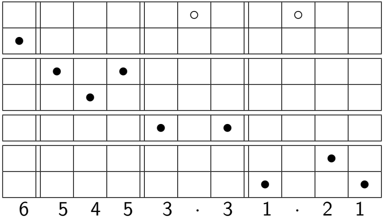


6 5 4 5 3 · 3 1 · 2 1

Squelette rouge de 8767595394312

$\mathrm{F}_{\color{red}{\mathrm{R}}\mathrm{ske}}(8767595394312) =$
$\mathrm{T}_{\color{red}{\mathrm{R}}\mathrm{ske}}(65453731721)\,\mathrm{T}_{\color{red}{\mathrm{R}}\mathrm{ske}}(12)$

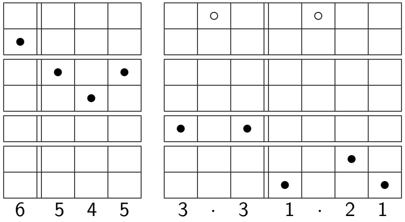
Factorisation en descentes globales



Squelette rouge de 8767595394312

$F_{\textcolor{red}{R}\mathrm{ske}}(8767595394312) = T_{\textcolor{red}{R}\mathrm{ske}}(65453731721)\, T_{\textcolor{red}{R}\mathrm{ske}}(12)$

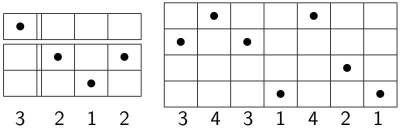
Distinction de deux groupes de facteurs



Squelette rouge de 8767595394312

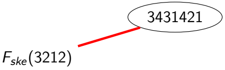
$F_{\text{Rske}}(8767595394312) = T_{\text{Rske}}(65453731721)\, T_{\text{Rske}}(12)$

Remise des lettres de valeur max + tassement



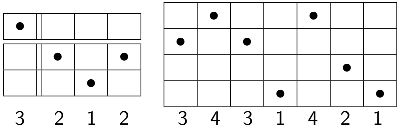
Squelette rouge de 8767595394312

$F_{\color{red}{R}ske}(8767595394312) =$
$T_{\color{red}{R}ske}(65453731721)\ T_{\color{red}{R}ske}(12) =$



$T_{\color{red}{R}ske}(12)$

3431421 est
Irréductible rouge



Irréductible rouge

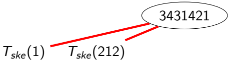
Un mot tassé *w* est **rouge irréductible** si il n'est pas décomposable par cet algorithme.

Squelette rouge de 8767595394312

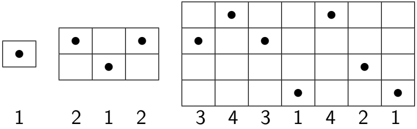
$\mathrm{F}_{\color{red}{\mathrm{R}}\mathrm{ske}}(8767595394312) =$

On boucle

$\mathrm{T}_{\color{red}{\mathrm{R}}\mathrm{ske}}(65453731721)\ \mathrm{T}_{\color{red}{\mathrm{R}}\mathrm{ske}}(12) =$



$\mathrm{T}_{\color{red}{\mathrm{R}}\mathrm{ske}}(12)$

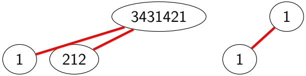


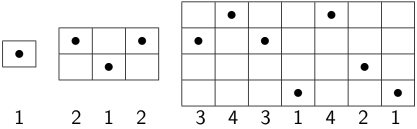
Irréductible rouge

Un mot tassé w est **rouge irréductible** si il n'est pas décomposable par cet algorithme.

Squelette rouge de 8767595394312

$F_{\color{red}{R}\text{ske}}(8767595394312) = \quad ((1/212) \blacktriangleright 3431421)/(1 \blacktriangleright 1)$



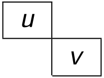


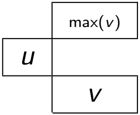
Irréductible rouge

Un mot tassé w est **rouge irréductible** si il n'est pas décomposable par cet algorithme.

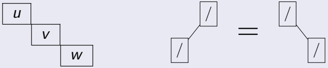
$\forall n, \mathit{RougeIrréductible}_n = \mathrm{TPrim}_n.$

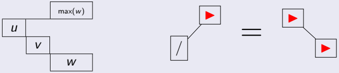
Operade (isomorphe à skew-duplicial)

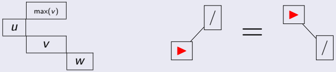
$u/v =$ 

$u \blacktriangleright v =$ 

Relations

- $(u/v)/w = u/(v/w)$ 

- $(u/v)\blacktriangleright w = u\blacktriangleright(v\blacktriangleright w)$ 

- $(u\blacktriangleright v)/w = u\blacktriangleright(v/w)$ 

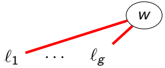
Dimensions comptés par Catalan + description combinatoire du dual de Koszul

Le début de la base $\mathbb{P}$

$$\mathbb{P}_{\textcircled{1}} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$

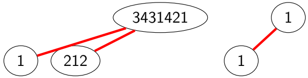


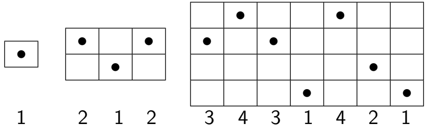
$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle.$$

Forêt rouge de 8767595394312

La partie droite!

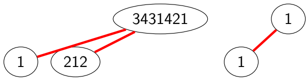
$F_{\color{red}R}(8767595394312) =$



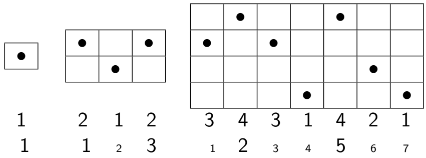


Forêt rouge de 8767595394312

$F_{\color{red}R}(8767595394312) =$



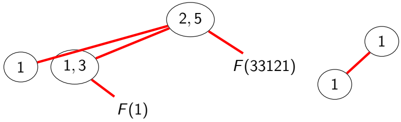
Positions des max

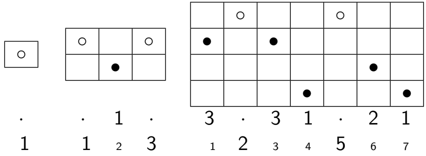


Forêt rouge de 8767595394312

Fils droits

$F_{\color{red}R}(8767595394312) =$

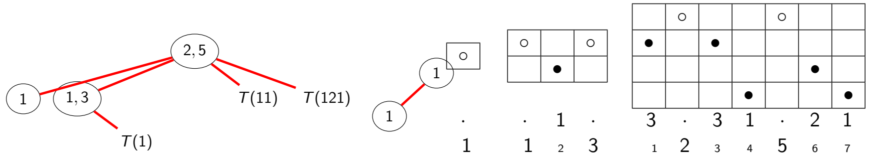




Forêt rouge de 8767595394312

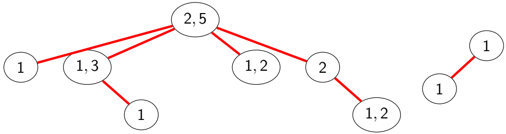
On reboucle

$\mathrm{F}_{\mathrm{R}}(8767595394312) =$



Forêt rouge de 8767595394312

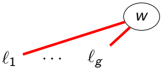
$F_{R}(8767595394312) =$



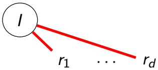
La base $\mathbb{P}$

$$\mathbb{P}_{1} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$ 

$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$

$\mathbb{P}$ 

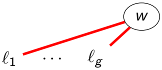
$$:= \Phi_I(\mathbb{P}_{r_1,\dots,r_d}).$$

La base $\mathbb{P}$

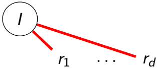
$\mathbb{P}$ 

$:= \mathbb{R}_1,$

$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

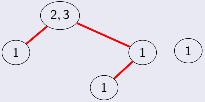
$\mathbb{P}$ 

$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$

$\mathbb{P}$ 

$:= \Phi_I(\mathbb{P}_{r_1,\dots,r_d}).$

Exemple

$\mathbb{P}$ 

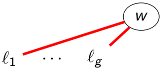
$$= \mathbb{R}_{235541} - \mathbb{R}_{245531} - \mathbb{R}_{244531} - \mathbb{R}_{245431} - \mathbb{R}_{254431} + \mathbb{R}_{325541} - \mathbb{R}_{425531} - \mathbb{R}_{524431} + \mathbb{R}_{352541} - \mathbb{R}_{452531} + \mathbb{R}_{355241} - \mathbb{R}_{455231} + \mathbb{R}_{344521} + \mathbb{R}_{345421} + \mathbb{R}_{354421} + \mathbb{R}_{534421}$$

La base $\mathbb{P}$

$$\mathbb{P}_{1} := \mathbb{R}_1,$$

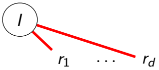
$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$



$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$

$\mathbb{P}$



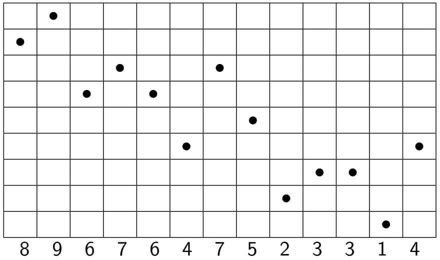
$$:= \Phi_l(\mathbb{P}_{r_1,\dots,r_d}).$$

Théorème [M.]

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_{R n}}$ est une base de **WQSym**$_n$,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_{R} n}$ est une base de Prim$_n$,
- $(\mathbb{P}_t)_{t \in \mathfrak{N}_{R n}}$ est une base de TPrim$_n$.

Squelette bleu de 8967647523314

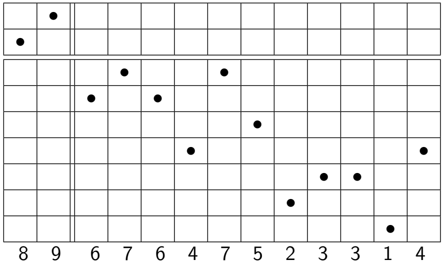
$F_{\color{blue}{B}\text{ske}}(8967647523314)$



Squelette bleu de 8967647523314

$F_{\color{blue}{B}\mathrm{ske}}(8967647523314)$

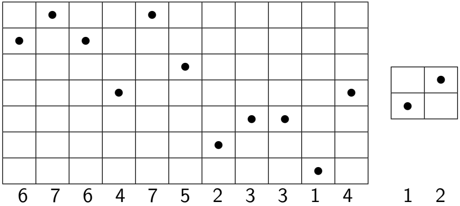
Factorisation en descentes globales



Squelette bleu de 8967647523314

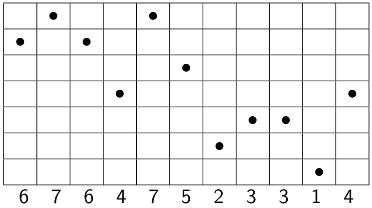
$\mathrm{F}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(8967647523314) =$
$\mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(67647523314)\,\mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(12)$

Factorisation en descentes globales
$+$ tassement $+$ échange



Squelette bleu de 8967647523314

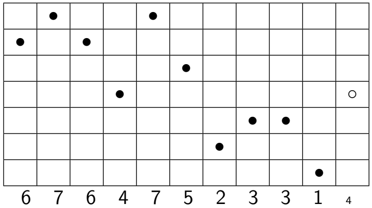
$$\mathsf{F}_{\color{blue}{\mathsf{B}}\mathsf{ske}}(8967647523314) = \mathsf{T}_{\color{blue}{\mathsf{B}}\mathsf{ske}}(67647523314)\,\mathsf{T}_{\color{blue}{\mathsf{B}}\mathsf{ske}}(12)$$



Squelette bleu de 8967647523314

$$F_{\mathrm{Bske}}(8967647523314) = T_{\mathrm{Bske}}(67647523314)\, T_{\mathrm{Bske}}(12)$$

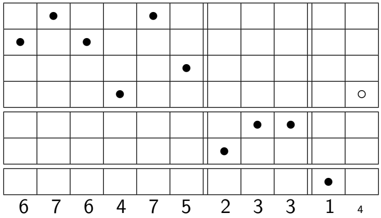
Retrait de la dernière lettre



Squelette bleu de 8967647523314

$F_{\color{blue}{B}\mathrm{ske}}(8967647523314) =$
$T_{\color{blue}{B}\mathrm{ske}}(67647523314)\, T_{\color{blue}{B}\mathrm{ske}}(12)$

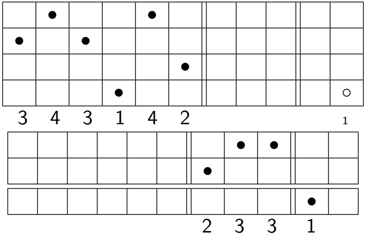
Factorisation en descentes globales



Squelette bleu de 8967647523314

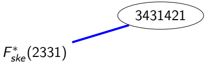
$\mathrm{F}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(8967647523314) = \mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(67647523314)\,\mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(12)$

Distinction de deux groupes de facteurs



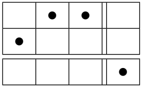
Squelette bleu de 8967647523314

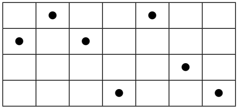
$\mathrm{F}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(8967647523314) =$
$\mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(67647523314)\,\mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(12) =$



$\mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(12)$

3431421 est Irréductible bleu





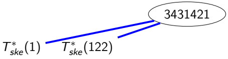
Irréductible bleu

Un mot tassé w est **bleu irréductible** si il n'est pas décomposable par cet algorithme.

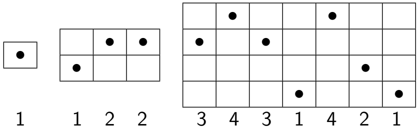
Squelette bleu de 8967647523314

$\mathrm{F}_{\mathrm{Bske}}(8967647523314) =$ On boucle

$\mathrm{T}_{\mathrm{Bske}}(67647523314)\ \mathrm{T}_{\mathrm{Bske}}(12) =$



$\mathrm{T}_{\mathrm{Bske}}(12)$

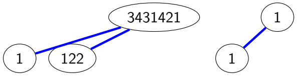


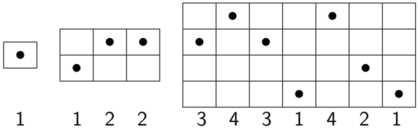
Irréductible bleu

Un mot tassé w est **bleu irréductible** si il n'est pas décomposable par cet algorithme.

Squelette bleu de 8967647523314

$\mathrm{F}_{\mathrm{Bske}}(8967647523314) = \quad ((1/122)\blacktriangle 3431421)/(1\blacktriangle 1)$





Irréductible bleu

Un mot tassé w est **bleu irréductible** si il n'est pas décomposable par cet algorithme.

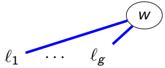
$\forall n, BleuIrréductible_n = RougeIrréductible_n = \mathrm{TPrim}_n.$

Le début de la base $\mathbb{O}$

$$\mathbb{O}_{1} := \mathbb{Q}_1,$$

$$\mathbb{O}_{t_1,\dots,t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

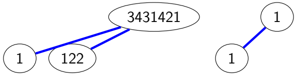
$\mathbb{O}$



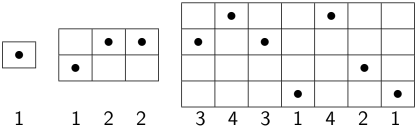
$$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle.$$

Forêt bleu de 8967647523314

$$F_{B}(8967647523314) =$$

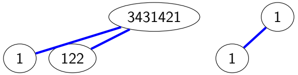


La partie droite!

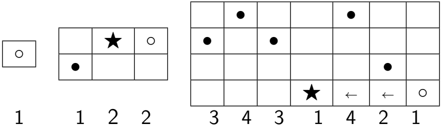


Forêt bleu de 8967647523314

$\mathsf{F}_{\color{blue}\mathsf{B}}(8967647523314) =$



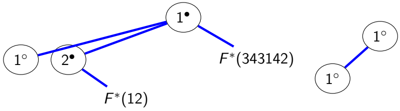
La dernière lettre est-elle présente dans le reste du mot ?

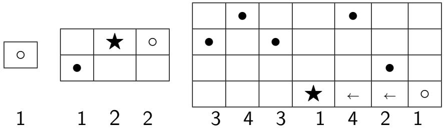


Forêt bleu de 8967647523314

Fils droits

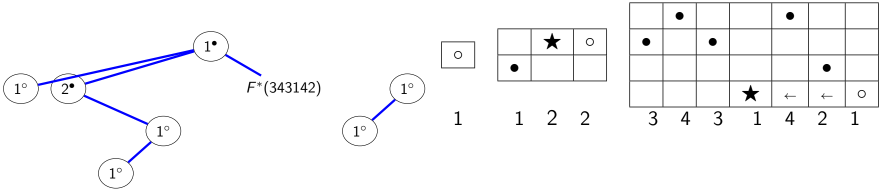
$F_{\mathrm{B}}(8967647523314) =$





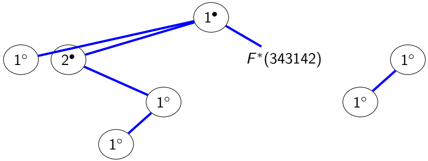
Forêt bleu de 8967647523314

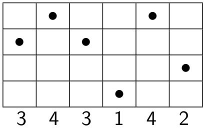
$\mathrm{F}_{\mathrm{B}}(8967647523314) =$



Forêt bleu de 8967647523314

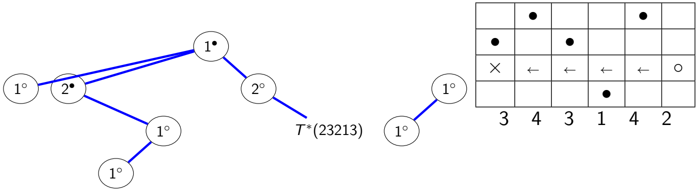
$\mathsf{F}_{\color{blue}{\mathsf{B}}}(8967647523314) =$





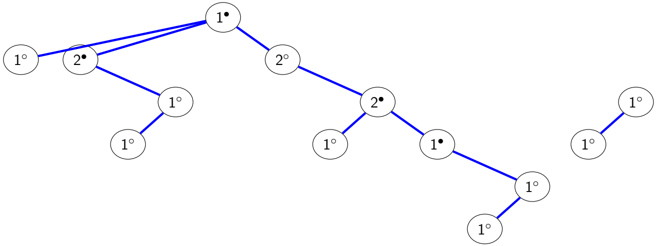
Forêt bleu de 8967647523314

$\mathsf{F}_{\color{blue}{\mathsf{B}}}(8967647523314) =$



Forêt bleu de 8967647523314

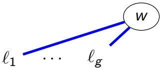
$$\mathsf{F}_{\color{blue}{\mathsf{B}}}(8967647523314) =$$



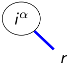
La base $\mathbb{O}$

$$\mathbb{O}_{1} := \mathbb{Q}_1,$$

$$\mathbb{O}_{t_1,\dots,t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$\mathbb{O}$ 

$$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$$

$\mathbb{O}$ 

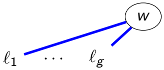
$$:= \Psi_i^\alpha(\mathbb{O}_r).$$

La base $\mathbb{O}$

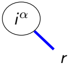
$\mathbb{O}$ 

$:= \mathbb{Q}_1,$

$$\mathbb{O}_{t_1,\dots,t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

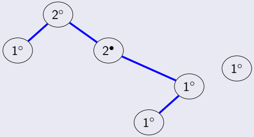
$\mathbb{O}$ 

$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$

$\mathbb{O}$ 

$:= \Psi_i^\alpha(\mathbb{O}_r).$

Exemple

$\mathbb{O}$ 

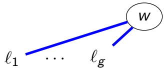
$$= \mathbb{Q}_{531442} + \mathbb{Q}_{521443} + \mathbb{Q}_{512443} - \mathbb{Q}_{534142} - \mathbb{Q}_{524143} - \mathbb{Q}_{514243} - \mathbb{Q}_{514432} - \mathbb{Q}_{524431} - \mathbb{Q}_{514423} + \mathbb{Q}_{541432} + \mathbb{Q}_{542431} + \mathbb{Q}_{541423}$$

La base $\mathbb{O}$

$$\mathbb{O}_{1} := \mathbb{Q}_1,$$

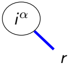
$$\mathbb{O}_{t_1,\dots,t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$\mathbb{O}$



$$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$$

$\mathbb{O}$



$$:= \Psi_i^\alpha(\mathbb{O}_r).$$

Théorème [M.]

- $(\mathbb{O}_f)_{f \in \mathfrak{F}_{B_n}}$ est une base de **WQSym**$_n^*$,
- $(\mathbb{O}_t)_{t \in \mathfrak{T}_{B_n}}$ est une base de Prim$_n^*$,
- $(\mathbb{O}_t)_{t \in \mathfrak{N}_{B_n}}$ est une base de TPrim$_n^*$.

Théorèmes [M.]

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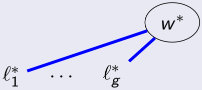
- $(\mathbb{O}_f)_{f\in\mathfrak{F}_{Bn}}$ base de $\mathbf{WQSym}_n^*$,
- $(\mathbb{O}_t)_{t\in\mathfrak{T}_{Bn}}$ base de Prim_n^*,
- $(\mathbb{O}_t)_{t\in\mathfrak{N}_{Bn}}$ base de TPrim_n^*.

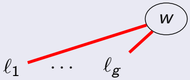
Théorème[M.]

- $(\mathbb{P}_f)_{f\in\mathfrak{F}_{Rn}}$ base de $\mathbf{WQSym}_n$,
- $(\mathbb{P}_t)_{t\in\mathfrak{T}_{Rn}}$ base de Prim_n,
- $(\mathbb{P}_t)_{t\in\mathfrak{N}_{Rn}}$ base de TPrim_n.

Rigidité

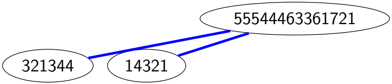


$\forall$ bijection entre les mots irréductibles bleus et rouges, recoloration des squelettes



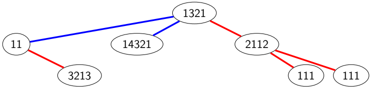
Forêts bicolores à travers un exemple

$\mathrm{T}_{\color{blue}{\mathrm{B}}\mathrm{ske}}(DDDCCCEBBE9FA587653213449) =$



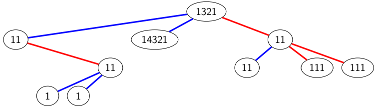
Forêts bicolores à travers un exemple

$\mathrm{T}_{\textcolor{blue}{B}\textcolor{red}{R}}(DDDCCCEBBE9FA587653213449)$



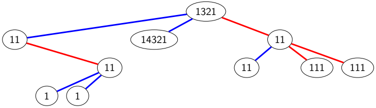
Forêts bicolores à travers un exemple

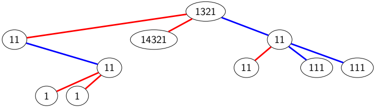
$\mathrm{T}_{\color{blue}{\mathrm{B}}\color{red}{\mathrm{R}}}(DDDCCCEBBE9FA587653213449) =$



Forêts bicolores à travers un exemple

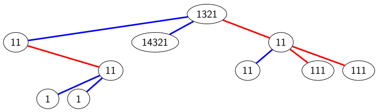
$T_{BR}(DDDCCCEBBE9FA587653213449) =$



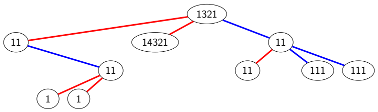


Forêts bicolores à travers un exemple

$\mathrm{T}_{\mathrm{BR}}(DDDCCCEBBE9FA587653213449) =$



$\mathrm{T}_{\mathrm{RB}}(DCBDDE7A9875F633422211145) =$



Théorèmes [M.]

Théorème [M.]

- $(\mathbb{O}_f)_{f \in \mathfrak{F}_{Bn}}$ base de $\mathbf{WQSym}_n^*$,
- $(\mathbb{O}_t)_{t \in \mathfrak{T}_{Bn}}$ base de Prim_n^*,
- $(\mathbb{O}_t)_{t \in \mathfrak{N}_{Bn}}$ base de TPrim_n^*.

Théorème [M.]

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_{Rn}}$ base de $\mathbf{WQSym}_n$,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_{Rn}}$ base de Prim_n,
- $(\mathbb{P}_t)_{t \in \mathfrak{N}_{Rn}}$ base de TPrim_n.

Bijection [M.]

Involution grâce aux forêts bicolores.
Isomorphisme bidendriforme entre **WQSym** et $\mathbf{WQSym}^*$.