

A bidendriform automorphism of WQSym

Seminar at York University

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Examples of Hopf algebras

- Planar binary trees, **PBT**, Loday-Ronco
- Non-commutative symmetric functions, **Sym**
- Quasi-symmetric functions, *QSym*
- Permutations, **FQSym**, Malvenuto-Reutenauer
- Packed words, **WQSym**, Hivert

Packed words

Definition

A word over the alphabet $\mathbb{N}_{>0}$ is packed if all the letters from 1 to its maximum m appears at least once.

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Packed words of size 0, 1, 2 and 3

- ϵ

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Packed words of size 0, 1, 2 and 3

- ϵ
- 1
- 12 21 11

Packed words

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Packed words of size 0, 1, 2 and 3

- ϵ
- 1
- 12 21 11
- 123 132 213 231 312 321
122 212 221 112 121 211 111

Packed words

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Packed words of size 0, 1, 2 and 3

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- 1
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- 123 132 213 231 312 321
 122 212 221 112 121 211 111

Packed words of size n [OEIS A000670]

n	1	2	3	4	5	6	7	8
PW_n	1	3	13	75	541	4683	47293	545835



Paking

Example

$24154 \notin \mathbf{PW}$

Paking

Example

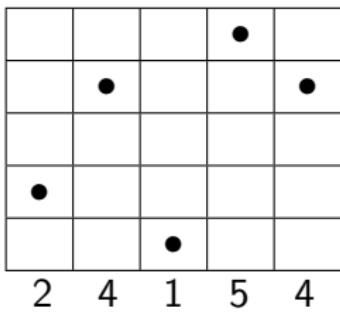
$24154 \notin \mathbf{PW}$ but $pack(24154) = 23143 \in \mathbf{PW}$

Paking

Example

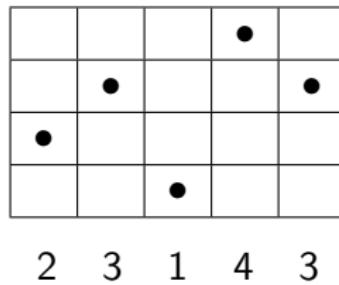
$24154 \notin \mathbf{PW}$ but $\text{pack}(24154) = 23143 \in \mathbf{PW}$

One representation : #rows \leq #columns



remove empty lines

\rightarrow pack \rightarrow



Hopf algebra

Example

WQSym

- $3112 + 212 - 3 \cdot 212341 - \frac{5}{3} \cdot 111$

Hopf algebra

Example

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$

Hopf algebra

Example

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
- $\mathbb{R}_{12}\mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$

Hopf algebra

Example

WQSym

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- $\Delta(\mathbb{R}_{24231}) = \mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_1 + \mathbb{R}_{24231} \otimes \mathbb{R}_\epsilon$

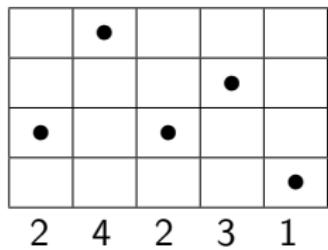
Hopf algebra

Example

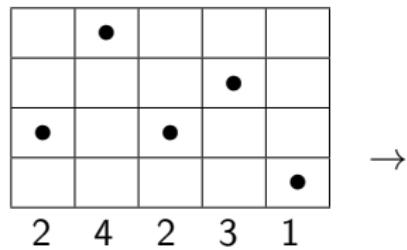
WQSym

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-
- unitary associative product \cdot
 - counitary coassociative coproduct Δ
 - Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

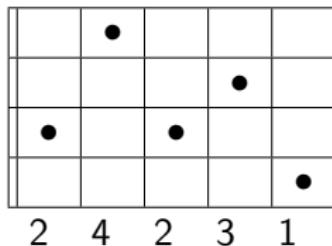
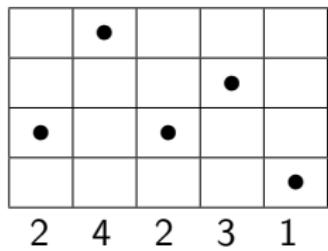
Deconcatenation (reduced)



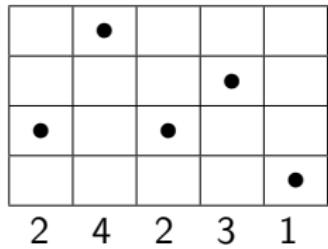
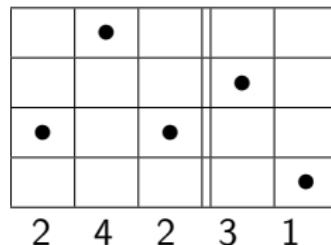
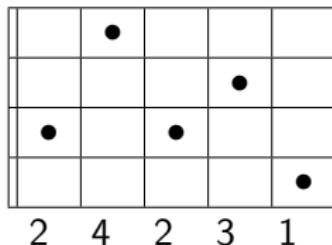
Deconcatenation (reduced)



Deconcatenation (reduced)



Deconcatenation (reduced)

 \rightarrow 

Deconcatenation (reduced)

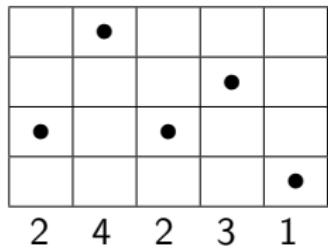
		•				
				•		
•			•			
						•
2	4	2	3	1		



		•				
				•		
•			•			
						•
2	4	2	3	1		

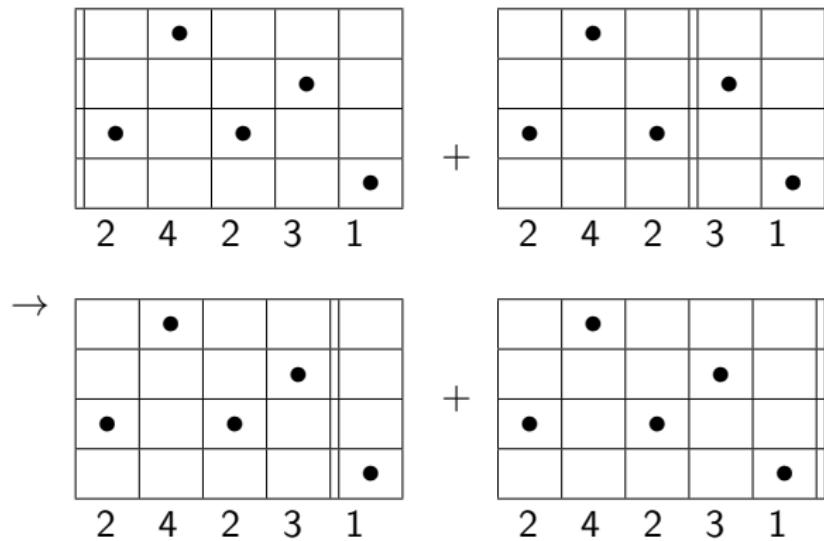
		•				
				•		
•			•			
						•
2	4	2	3	1		

Deconcatenation (reduced)

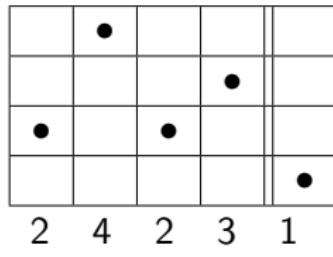


$$\begin{array}{c}
 \begin{array}{|c|c|c|c|c|c|} \hline & & \bullet & & & \\ \hline & & & & \bullet & \\ \hline & \bullet & & \bullet & & \\ \hline & & & & & \bullet \\ \hline 2 & 4 & 2 & 3 & 1 & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline & & \bullet & & & \\ \hline & & & & \bullet & \\ \hline & \bullet & & \bullet & & \\ \hline & & & & & \bullet \\ \hline 2 & 4 & 2 & 3 & 1 & \\ \hline \end{array} \\
 \xrightarrow{} \begin{array}{|c|c|c|c|c|c|} \hline & & \bullet & & & \\ \hline & & & & \bullet & \\ \hline & \bullet & & \bullet & & \\ \hline & & & & & \bullet \\ \hline 2 & 4 & 2 & 3 & 1 & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline & & \bullet & & & \\ \hline & & & & \bullet & \\ \hline & \bullet & & \bullet & & \\ \hline & & & & & \bullet \\ \hline 2 & 4 & 2 & 3 & 1 & \\ \hline \end{array}
 \end{array}$$

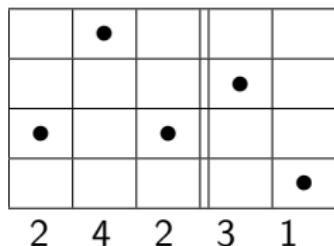
Deconcatenation (reduced)

 \mathbb{R}_{24231} 

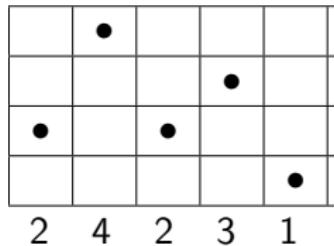
Deconcatenation (reduced)

 \mathbb{R}_{24231} $\Delta \rightarrow$  $\mathbb{R}_\epsilon \otimes \mathbb{R}_{24231}$

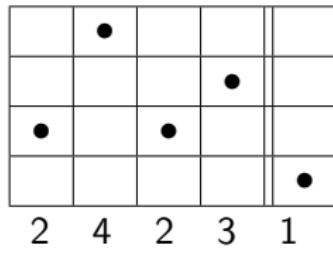
+



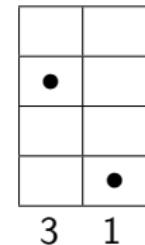
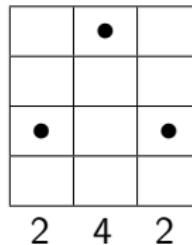
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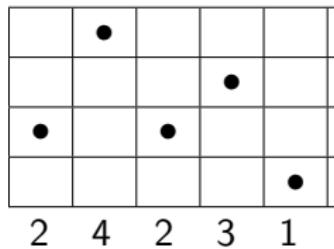
Deconcatenation (reduced)

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+



+



Deconcatenation (reduced)

$$\begin{aligned}
 & \mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} \\
 + & \begin{array}{c} \text{+} \\ \begin{array}{ccc|cc} & & \bullet & & \\ \bullet & & & & \bullet \\ & & & & \\ \hline 1 & 2 & 1 & & \\ & 2 & 1 & & \end{array} \end{array} \\
 \mathbb{R}_{24231} & \xrightarrow{\Delta} \begin{array}{c} \Delta \\ \rightarrow \end{array} \begin{array}{c} \begin{array}{ccccc|c} & & \bullet & & & & \\ & & & & & & \\ \bullet & & & \bullet & & & \\ & & & & \bullet & & \\ & & & & & & \\ \hline 2 & 4 & 2 & 3 & 1 & & \bullet \\ & & & & & & \end{array} \\ + \begin{array}{ccccc|c} & & \bullet & & & & \\ & & & & & & \\ \bullet & & & & \bullet & & \\ & & & & & & \\ & & & & & & \\ \hline 2 & 4 & 2 & 3 & 1 & & \bullet \\ & & & & & & \bullet \end{array} \end{array}
 \end{aligned}$$

Deconcatenation (reduced)

$$\mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21}$$

 \mathbb{R}_{24231} Δ
 \rightarrow

		•				
				•		
•			•			
					•	
2	4	2	3	1		

+

	•					
				•		
•			•			
					•	
2	4	2	3	1		

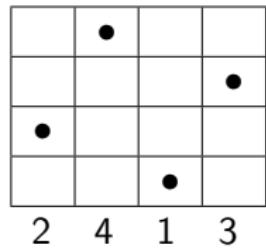
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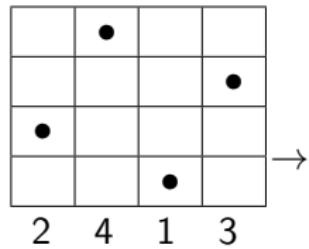
$$\mathbb{R}_{24231} \xrightarrow{\Delta}$$

$$\mathbb{R}_{1312} \otimes \mathbb{R}_1 + \mathbb{R}_{24231} \otimes \mathbb{R}_\epsilon$$

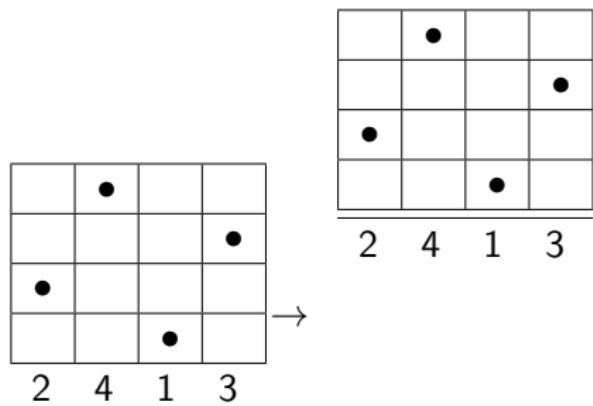
Horizontal disassembly



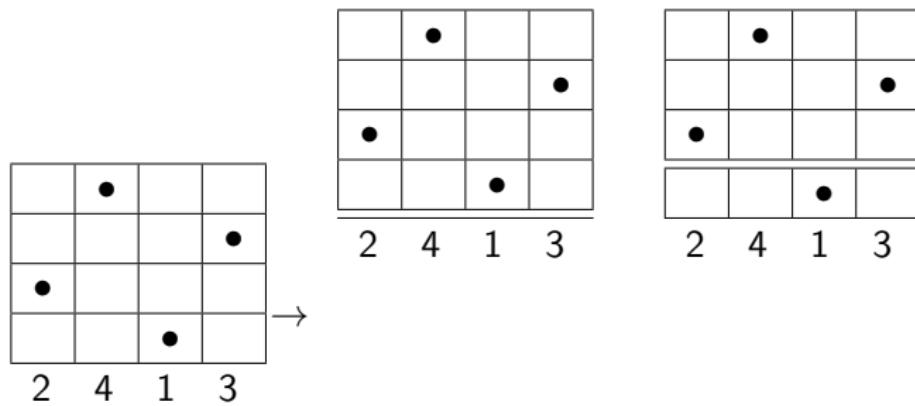
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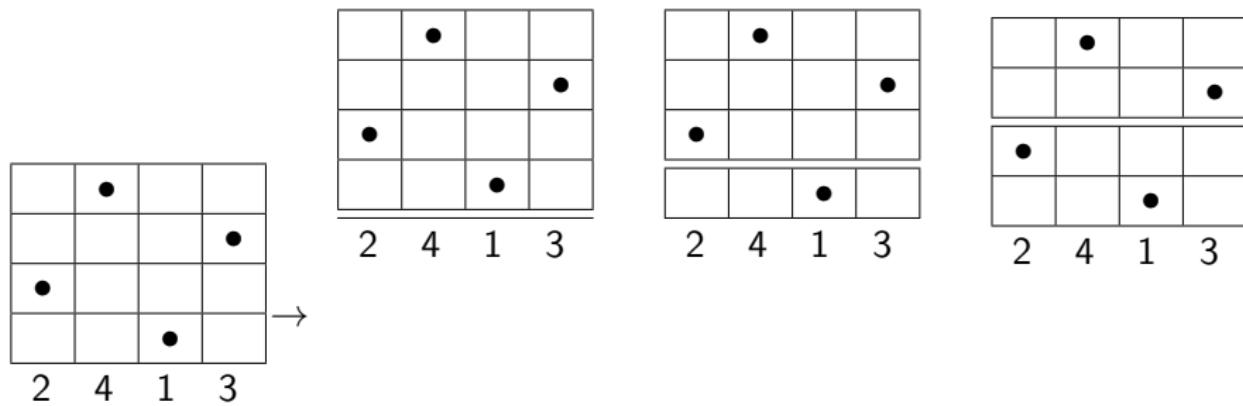
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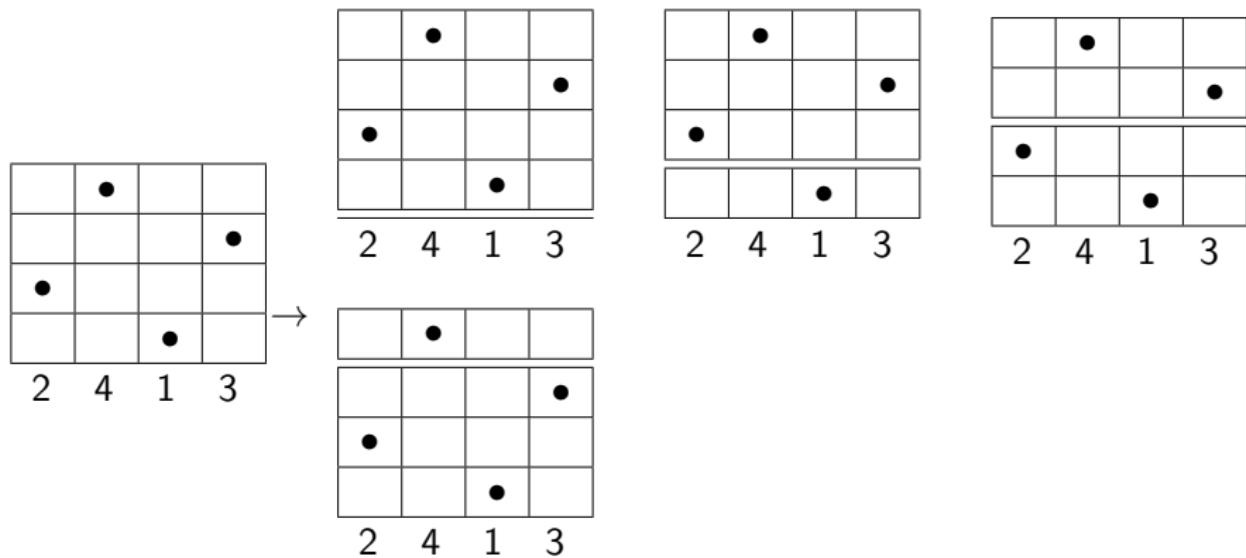
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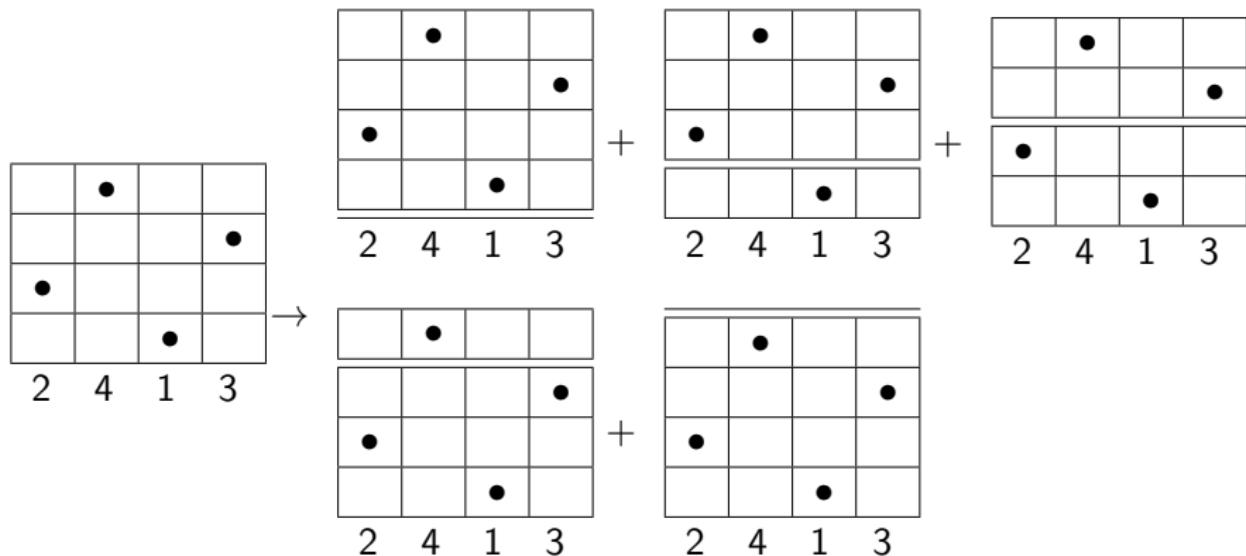
Horizontal disassembly



Horizontal disassembly



Horizontal disassembly



Horizontal disassembly

 Q_{2413}

$$\begin{array}{c} \Delta \\ \rightarrow \end{array} \begin{array}{c} \begin{array}{cccc} & \bullet & & \\ & & & \\ & & & \\ & & & \\ \bullet & & & \\ & & & \\ & & & \\ & & & \\ & & & \bullet \end{array} + \begin{array}{cccc} & \bullet & & \\ & & & \\ & & & \\ & & & \\ \bullet & & & \\ & & & \\ & & & \\ & & & \\ & & & \bullet \end{array} + \begin{array}{cccc} & & \bullet & \\ & & & \\ & & & \\ & & & \\ & & & \\ \bullet & & & \\ & & & \\ & & & \\ & & & \bullet \end{array} \\ 2 \quad 4 \quad 1 \quad 3 \qquad 2 \quad 4 \quad 1 \quad 3 \qquad 2 \quad 4 \quad 1 \quad 3 \end{array}$$

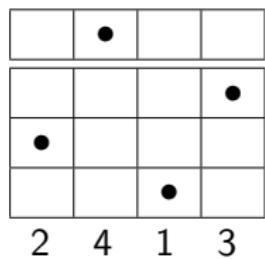
Horizontal disassembly

$$\begin{array}{c}
 Q_\epsilon \otimes Q_{2413+} \\
 \Delta \rightarrow \\
 Q_{2413}
 \end{array}
 + \begin{array}{c}
 \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \\ \hline \bullet & & & \\ \hline & & & \\ \hline & & \bullet & \\ \hline \end{array} \\
 2 \quad 4 \quad 1 \quad 3
 \end{array}
 + \begin{array}{c}
 \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \\ \hline \bullet & & & \\ \hline & & & \\ \hline & & \bullet & \\ \hline \end{array} \\
 2 \quad 4 \quad 1 \quad 3
 \end{array}
 + \begin{array}{c}
 \begin{array}{|c|c|c|c|} \hline & \bullet & & \\ \hline & & & \\ \hline \bullet & & & \\ \hline & & & \\ \hline & & \bullet & \\ \hline \end{array} \\
 2 \quad 4 \quad 1 \quad 3
 \end{array}$$

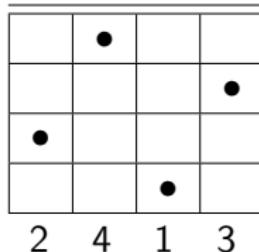
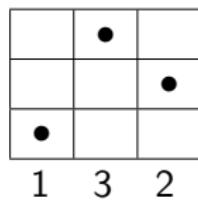
Horizontal disassembly

$$\begin{array}{c}
 Q_\epsilon \otimes Q_{2413+} \\
 \Delta \rightarrow
 \end{array}
 \begin{array}{c}
 \begin{array}{c}
 \begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline & & & \\ \hline & \bullet & & \\ \hline 1 & 3 & & 2 \\ \hline & & \bullet & \\ \hline & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & \bullet & \\ \hline & & & \\ \hline & \bullet & & \\ \hline 2 & 4 & & 3 \\ \hline & & \bullet & \\ \hline & & & 1 \\ \hline \end{array}
 \end{array} \\
 \begin{array}{c}
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 \end{array}
 \end{array}$$

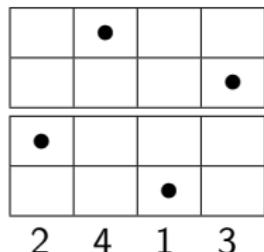
Horizontal disassembly

 Q_{2413}
 Δ
 \rightarrow


+


 $Q_\epsilon \otimes Q_{2413+}$


+



Horizontal disassembly

$$\begin{aligned}
 & Q_\epsilon \otimes Q_{2413} + Q_1 \otimes Q_{132} + \\
 & Q_{2413} \xrightarrow{\Delta} \begin{array}{c} \text{---} \\ | \\ \begin{array}{cccc} & \bullet & & \\ \hline & & & \\ & & & \bullet \\ \hline \bullet & & & \bullet \\ \hline & & \bullet & \\ \hline 2 & 4 & 1 & 3 \end{array} \end{array} + \begin{array}{c} \text{---} \\ | \\ \begin{array}{cccc} & \bullet & & \\ \hline & & & \\ & & & \bullet \\ \hline \bullet & & & \bullet \\ \hline & \bullet & & \\ \hline 2 & 4 & 1 & 3 \end{array} \end{array} + \begin{array}{c} \text{---} \\ | \\ \begin{array}{cccc} & \bullet & & \\ \hline & & & \\ & & & \bullet \\ \hline & & \bullet & \\ \hline & & & \bullet \\ \hline 2 & 4 & 1 & 3 \end{array} \end{array}
 \end{aligned}$$

Horizontal disassembly

$$\mathbb{Q}_\epsilon \otimes \mathbb{Q}_{2413} + \mathbb{Q}_1 \otimes \mathbb{Q}_{132} + \mathbb{Q}_{21} \otimes \mathbb{Q}_{21}$$

$$\begin{array}{ccc} \mathbb{Q}_{2413} & \xrightarrow{\Delta} & \\ \rightarrow & & \end{array}$$

$$\mathbb{Q}_{213} \otimes \mathbb{Q}_1 + \mathbb{Q}_{2413} \otimes \mathbb{Q}_\epsilon$$

Auto-duality

- \mathbb{R} and \mathbb{Q} bases of **WQSym** and **WQSym***

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- 2001 Duchamp-Hivert-Thibon conjecture the auto-duality of **WQSym**

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Auto-duality

- \mathbb{R} and \mathbb{Q} bases of **WQSym** and **WQSym***
- 2001 Duchamp-Hivert-Thibon conjecture the auto-duality of **WQSym**
- 2005 Foissy demonstrates the self-duality of bidendriform bialgebra (rigidity)
- No explicit isomorphism

Half coproducts

Example of left and right coproducts

- $\tilde{\Delta}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321} + \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_1$

Half coproducts

Example of left and right coproducts

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- $\Delta_{\prec}(\mathbb{R}_{2425531}) = \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_1$
- $\Delta_{\succ}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321}$

Half coproducts

Definitions

- $\Delta_{\prec}(\mathbb{R}_u) := \sum_{\substack{i=1 \\ \{u_1, \dots, u_i\} \cap \{u_{i+1}, \dots, u_n\} = \emptyset \\ u_k = \max(u)}}^{n-1} \mathbb{R}_{\text{pack}(u_1 \cdots u_i)} \otimes \mathbb{R}_{\text{pack}(u_{i+1} \cdots u_n)},$
- $\Delta_{\succ}(\mathbb{R}_u) := \sum_{\substack{i=1 \\ \{u_1, \dots, u_i\} \cap \{u_{i+1}, \dots, u_n\} = \emptyset \\ u_k = \max(u)}}^{k-1} \mathbb{R}_{\text{pack}(u_1 \cdots u_i)} \otimes \mathbb{R}_{\text{pack}(u_{i+1} \cdots u_n)}$

Example of left and right coproducts

- $\tilde{\Delta}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321} + \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_1$
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Bidendriform bialgebra

Definition

- Refinement of associativity and co-associativity
 - 3 and 3 equations

Bidendriform bialgebra

Definition

- Refinement of associativity and co-associativity
 - 3 and 3 equations
- Refinement of the Hopf relation
 - 4 equations

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 - 4 equations

Theorem [Foissy]

If A is a bidendriform bialgebra then A is freely generated by $T\text{Prim}(A)$ as a dendriform algebra.

Bidendriform bialgebra

Definition

- Refinement of associativity and co-associativity
 - 3 and 3 equations
- Refinement of the Hopf relation
 - 4 equations

Theorem [Foissy]

If A is a bidendriform bialgebra then A is freely generated by $T\text{Prim}(A)$ as a dendriform algebra.

Series

n	1	2	3	4	5	6	7	8
WQSym_n	1	3	13	75	541	4 683	47 293	545 835
$T\text{Prim}_n$	1	1	4	28	240	2 384	26 832	337 168

Bidendriform bialgebra

Definition

- Refinement of associativity and co-associativity
 - 3 and 3 equations
- Refinement of the Hopf relation
 - 4 equations

Theorem [Foissy]

If A is a bidendriform bialgebra then A is freely generated by $T\text{Prim}(A)$ as a dendriform algebra.

Corollary

WQSym is self-dual.

Definitions

Primitive element

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

Definitions

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P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_1$$

$$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_1$$

Definitions

Primitive element

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_1$$

$$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_1$$

Totally primitive element

P is a totally primitive element $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

Definitions

Primitive element

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_1$$

$$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_1$$

Totally primitive element

P is a totally primitive element $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

$$\tilde{\Delta}(\mathbb{R}_{12443}) = \mathbb{R}_{1233} \otimes \mathbb{R}_1 \quad \mathbb{R}_{12} \otimes \mathbb{R}_{221} + \mathbb{R}_1 \otimes \mathbb{R}_{1332}$$

$$\tilde{\Delta}(\mathbb{R}_{21443}) = \mathbb{R}_{2133} \otimes \mathbb{R}_1 \quad \mathbb{R}_{21} \otimes \mathbb{R}_{221} + \mathbb{R}_1 \otimes \mathbb{R}_{1332}$$

$$\tilde{\Delta}(\mathbb{R}_{23441}) = \mathbb{R}_{1233} \otimes \mathbb{R}_1 \quad \mathbb{R}_{12} \otimes \mathbb{R}_{221} + \mathbb{R}_1 \otimes \mathbb{R}_{2331}$$

$$\tilde{\Delta}(\mathbb{R}_{32441}) = \mathbb{R}_{2133} \otimes \mathbb{R}_1 \quad \mathbb{R}_{21} \otimes \mathbb{R}_{221} + \mathbb{R}_1 \otimes \mathbb{R}_{2331}$$

My goal

Explicit bidendriform isomorphism between **WQSym** and it's dual

My goal

Explicit bidendriform isomorphism between **WQSym** and it's dual



Explicit isomorphism between $T\text{Prim}(\mathbf{WQSym})$ and it's dual

My goal

Explicit bidendriform isomorphism between **WQSym** and it's dual



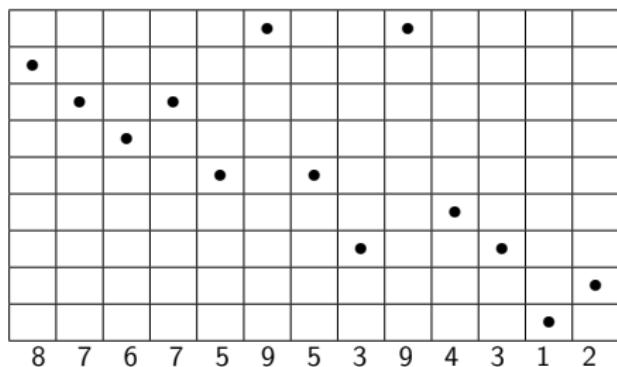
Explicit isomorphism between $T\text{Prim}(\mathbf{WQSym})$ and it's dual

Construction of two bases of totally primitive
(in **WQSym** and **WQSym** *)

Biplane Forests, representation of decompositions

	Skeletons	
	Paked trees	
	Bicolors trees	

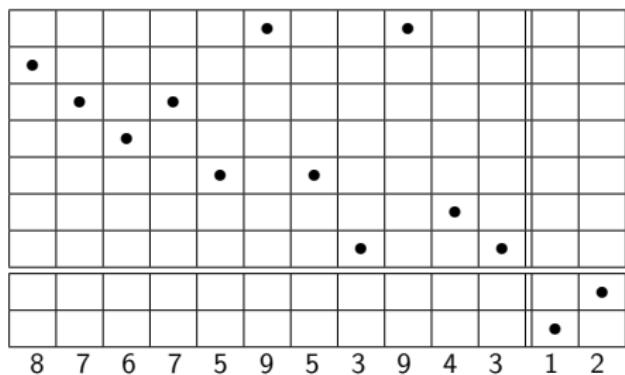
Red skeleton of 8767595394312

 $F_{ske}(8767595394312)$ 

Red skeleton of 8767595394312

 $F_{ske}(8767595394312)$

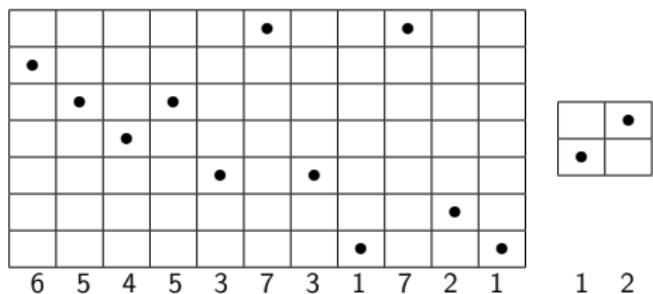
Global descents factorisation



Red skeleton of 8767595394312

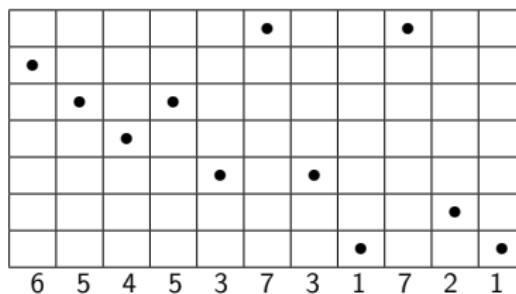
$$F_{ske}(8767595394312) = \\ T_{ske}(65453731721) T_{ske}(12)$$

Global descents factorisation +
packing



Red skeleton of 8767595394312

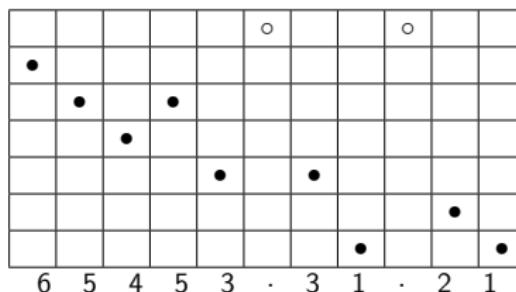
$$F_{ske}(8767595394312) = \\ T_{ske}(65453731721) T_{ske}(12)$$



Red skeleton of 8767595394312

$$\begin{aligned} F_{ske}(8767595394312) = \\ T_{ske}(65453731721) T_{ske}(12) \end{aligned}$$

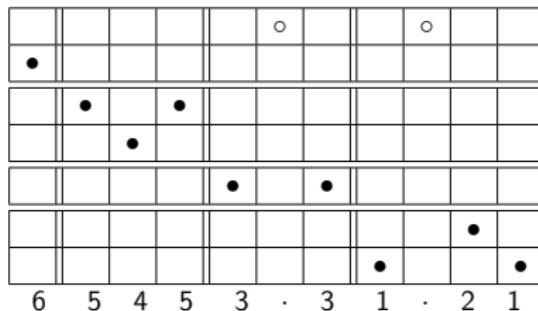
Remove all the occurrences of the maximal value



Red skeleton of 8767595394312

$$F_{ske}(8767595394312) = \\ T_{ske}(65453731721) T_{ske}(12)$$

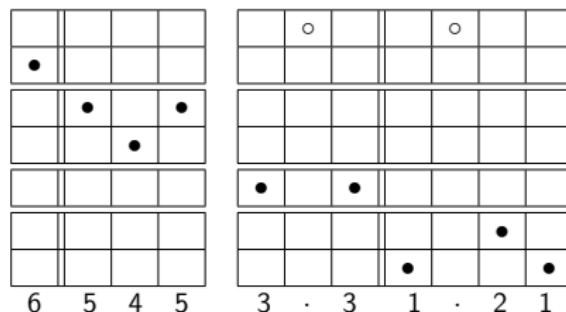
Global descents factorisation



Red skeleton of 8767595394312

$$F_{ske}(8767595394312) = \\ T_{ske}(65453731721) T_{ske}(12)$$

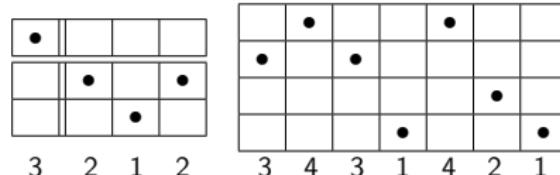
Distinction of two groups of factors



Red skeleton of 8767595394312

$$\begin{aligned} F_{ske}(8767595394312) = \\ T_{ske}(65453731721) T_{ske}(12) \end{aligned}$$

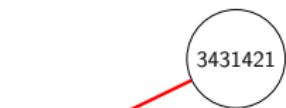
Reinsert the removed letters +
packing



Red skeleton of 8767595394312

$$F_{ske}(8767595394312) =$$

$$T_{ske}(65453731721) T_{ske}(12) =$$



$$F_{ske}(3212)$$

$$T_{ske}(12)$$

3431421 is
Red irreducible

•				
	•			•
		•		•

3 2 1 2

•						
•		•			•	
		•				•
			•			
				•		
					•	
						•

3 4 3 1 4 2 1

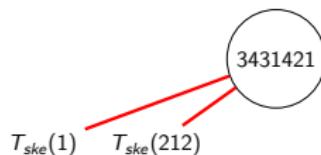
Red irreducible

A packed word w is **red irreducible** if it is not decomposable by this algorithm.

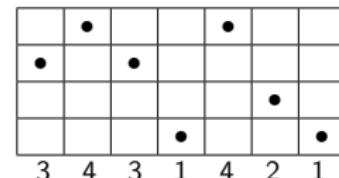
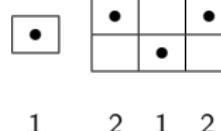
Red skeleton of 8767595394312

$F_{ske}(8767595394312) =$ Loop

$T_{ske}(65453731721) T_{ske}(12) =$



$T_{ske}(12)$

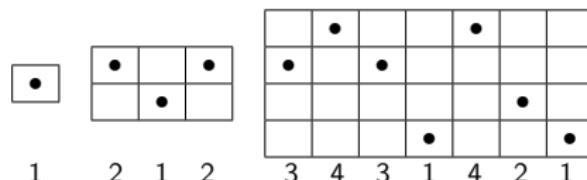
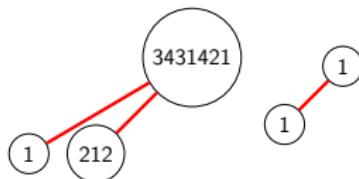


Red irreducible

A packed word w is **red irreducible** if it is not decomposable by this algorithm.

Red skeleton of 8767595394312

$F_{ske}(8767595394312) =$



Red irreducible

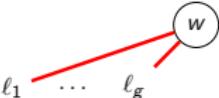
A packed word w is **red irreducible** if it is not decomposable by this algorithm.

$$\forall n, \# RedIrreducible_n = \dim(TPrim_n).$$

First part for basis \mathbb{P}

$$\mathbb{P}_{(1)} := \mathbb{R}_1,$$

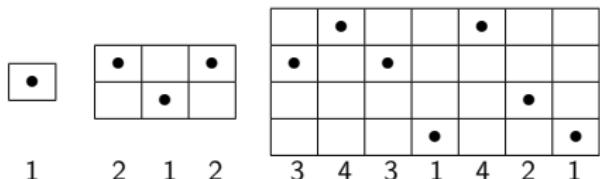
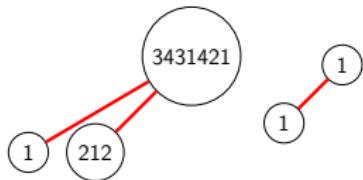
$$\mathbb{P}_{t_1, \dots, t_k} := (\dots (\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$$\mathbb{P} := \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle.$$
A diagram showing a circular node labeled 'w' at the top right. From its bottom left, several red lines radiate outwards to the left, each labeled with a symbol ℓ_1, \dots, ℓ_g .

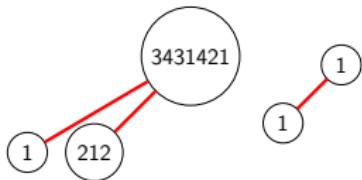
Red forest of 8767595394312

The right part!

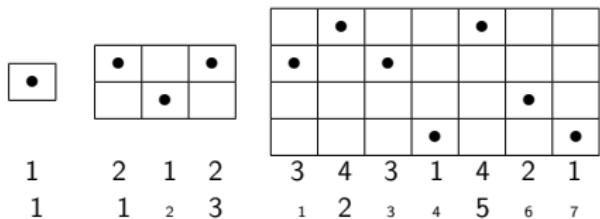
$$F(8767595394312) =$$



Red forest of 8767595394312

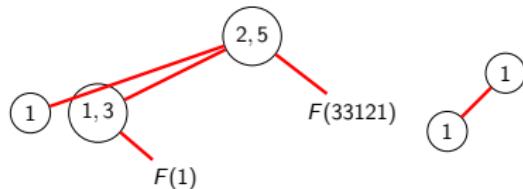
 $F(8767595394312) =$ 

Positions of max

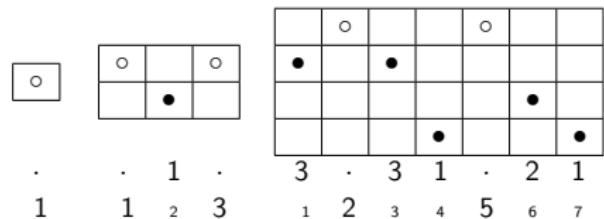


Red forest of 8767595394312

$$F(8767595394312) =$$



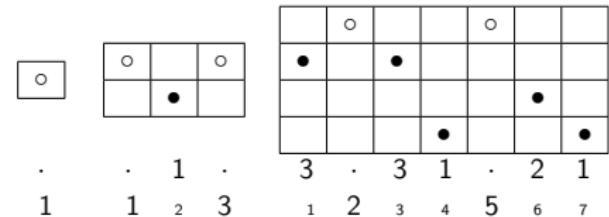
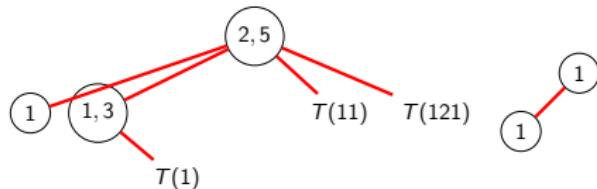
Right children



Red forest of 8767595394312

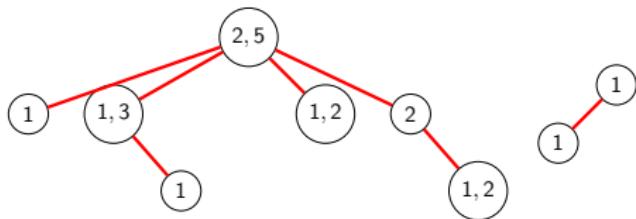
Loop again

$$F(8767595394312) =$$



Red forest of 8767595394312

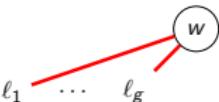
$$F(8767595394312) =$$



The basis \mathbb{P}

$$\mathbb{P}_{\textcircled{1}} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1, \dots, t_k} := (\dots (\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$$\mathbb{P} := \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$


$$\mathbb{P}_{\textcircled{I}} := \Phi_I(\mathbb{P}_{r_1, \dots, r_d}).$$


The basis \mathbb{P}

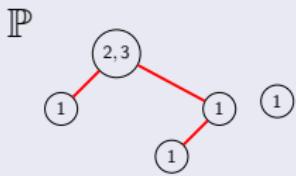
$$\mathbb{P}_{(1)} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1, \dots, t_k} := (\dots (\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$$\mathbb{P} := \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$

$$\mathbb{P}_{(I)} := \Phi_I(\mathbb{P}_{r_1, \dots, r_d}).$$

Example



$$\begin{aligned} \mathbb{P} = & \mathbb{R}_{235541} - \mathbb{R}_{245531} - \mathbb{R}_{244531} - \mathbb{R}_{245431} - \\ & \mathbb{R}_{254431} + \mathbb{R}_{325541} - \mathbb{R}_{425531} - \mathbb{R}_{524431} + \\ & \mathbb{R}_{352541} - \mathbb{R}_{452531} + \mathbb{R}_{355241} - \mathbb{R}_{455231} + \\ & \mathbb{R}_{344521} + \mathbb{R}_{345421} + \mathbb{R}_{354421} + \mathbb{R}_{534421} \end{aligned}$$



The basis \mathbb{P}

$$\mathbb{P}_{(1)} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1, \dots, t_k} := (\dots (\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$$\mathbb{P}_w := \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$

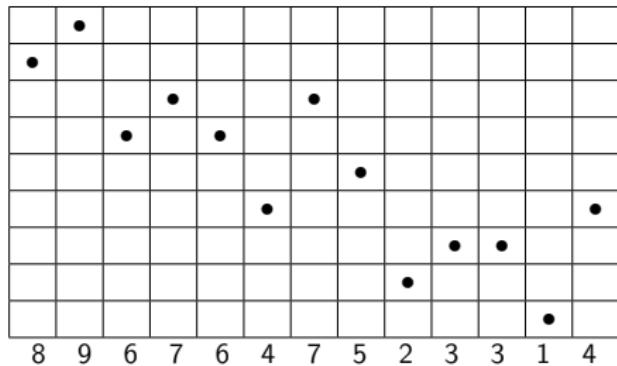
$$\mathbb{P}_I := \Phi_I(\mathbb{P}_{r_1, \dots, r_d}).$$

Theorem [M.]

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_n}$ is a basis of \mathbf{WQSym}_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_n}$ is a basis of Prim_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{P}_n}$ is a basis of TPrim_n .



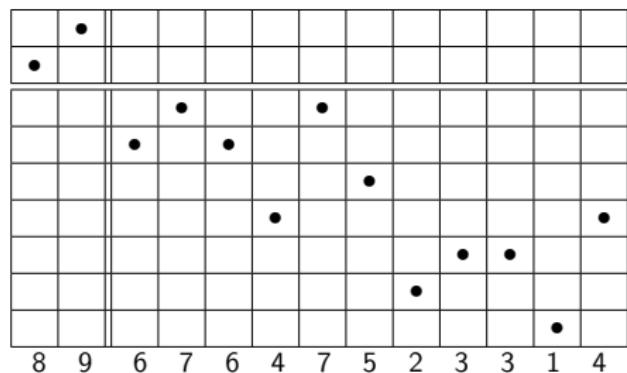
Blue skeleton of 8967647523314

 $F_{ske}^*(8967647523314)$ 

Blue skeleton of 8967647523314

 $F_{ske}^*(8967647523314)$

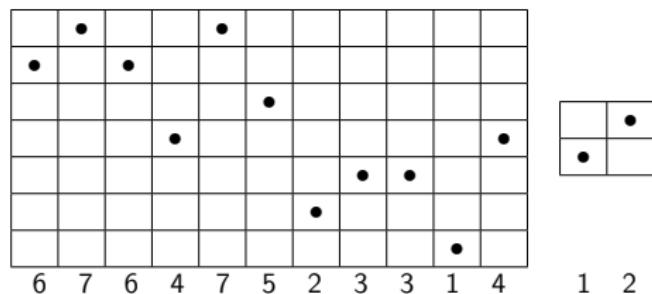
Global descents factorisation



Blue skeleton of 8967647523314

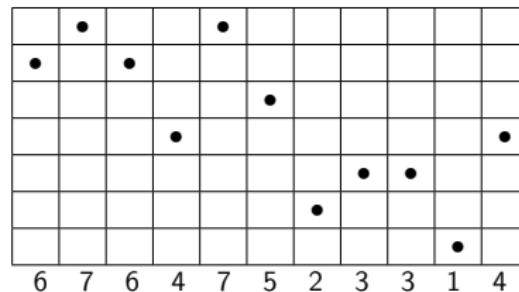
$$F_{ske}^*(8967647523314) =$$

Global descents factorisation + packing + swap



Blue skeleton of 8967647523314

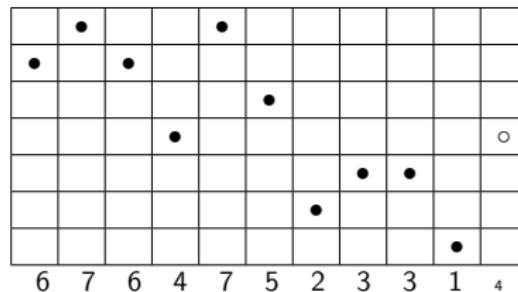
$$F_{ske}^*(8967647523314) = \\ T_{ske}^*(67647523314) T_{ske}^*(12)$$



Blue skeleton of 8967647523314

$$F_{ske}^*(8967647523314) = T_{ske}^*(67647523314) T_{ske}^*(12)$$

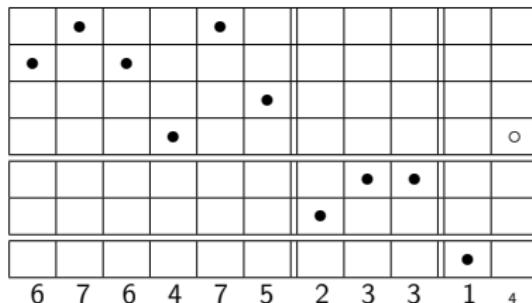
Remove of the last letter



Blue skeleton of 8967647523314

$$F_{ske}^*(8967647523314) = \\ T_{ske}^*(67647523314) T_{ske}^*(12)$$

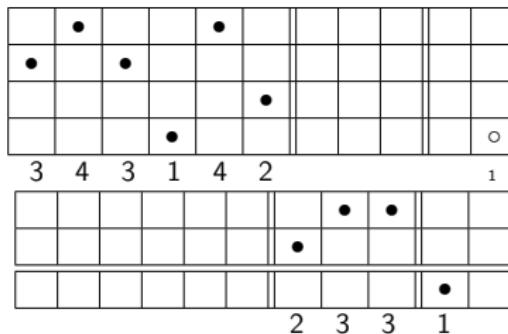
Global descents factorisation



Blue skeleton of 8967647523314

$$\begin{aligned} F_{ske}^*(8967647523314) = \\ T_{ske}^*(67647523314) T_{ske}^*(12) \end{aligned}$$

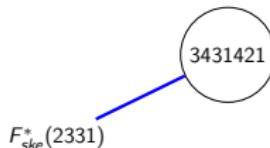
Distinction of two groups of factors



Blue skeleton of 8967647523314

$$F_{ske}^*(8967647523314) =$$

$$T_{ske}^*(67647523314) T_{ske}^*(12) =$$



3431421 is
Blue irreducible

		•	•		
•			•		
				•	

2 3 3 1

	•			•		
•			•			
					•	
				•		
3	4	3	1	4	2	1

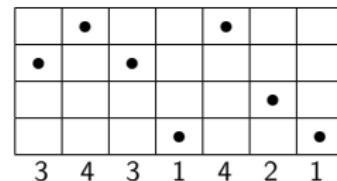
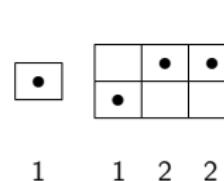
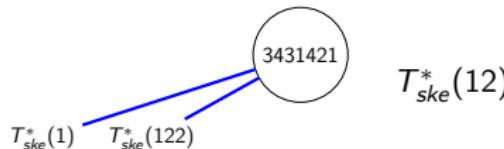
Blue irreducible

A packed word w is **blue irreducible** if it is not decomposable by this algorithm.

Blue skeleton of 8967647523314

$$F_{ske}^*(8967647523314) = \text{Loop}$$

$$T_{ske}^*(67647523314) T_{ske}^*(12) =$$

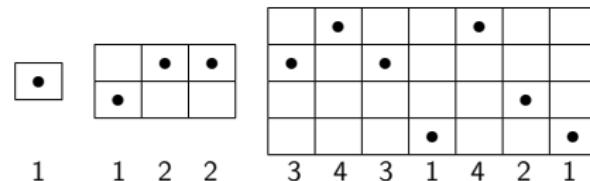
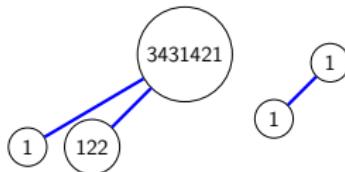


Blue irreducible

A packed word w is **blue irreducible** if it is not decomposable by this algorithm.

Blue skeleton of 8967647523314

$$F_{ske}^*(8967647523314) =$$



Blue irreducible

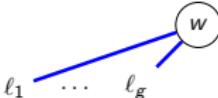
A packed word w is **blue irreducible** if it is not decomposable by this algorithm.

$$\forall n, \# \text{BlueIrreducible}_n = \# \text{RedIrreducible}_n = \dim(\text{TPrim}_n).$$

First part for basis \mathbb{O}

$$\mathbb{O}_{\textcircled{1}} := \mathbb{Q}_1,$$

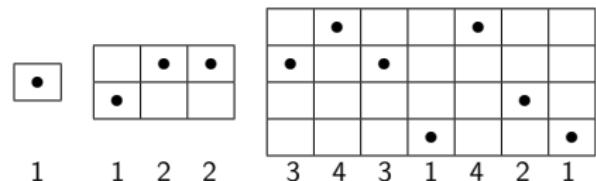
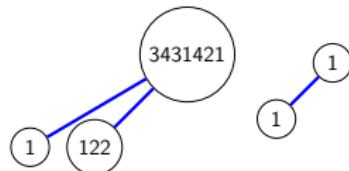
$$\mathbb{O}_{t_1, \dots, t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$$\mathbb{O} := \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle.$$
A diagram showing a node labeled 'w' enclosed in a circle. Below it, several blue lines radiate outwards, each labeled with a symbol from the sequence ℓ_1, \dots, ℓ_g .

Blue forest of 8967647523314

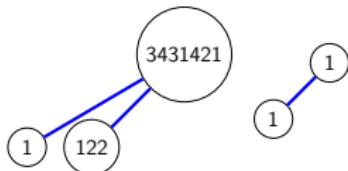
The right part!

$$F^*(8967647523314) =$$

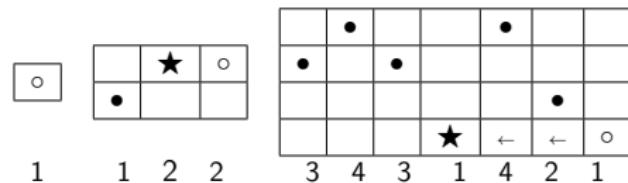


Blue forest of 8967647523314

$$F^*(8967647523314) =$$



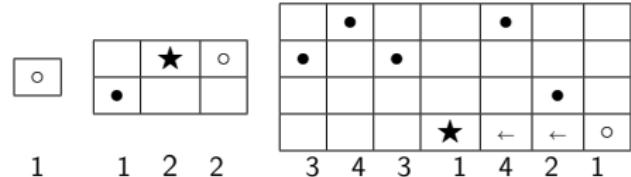
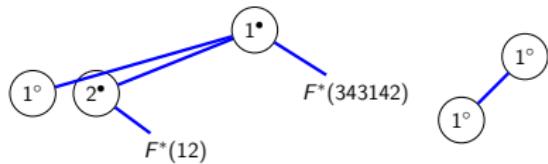
The last letter appears in the rest of the word?



Blue forest of 8967647523314

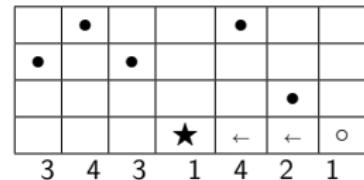
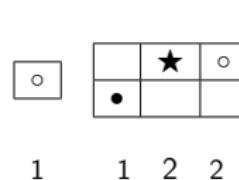
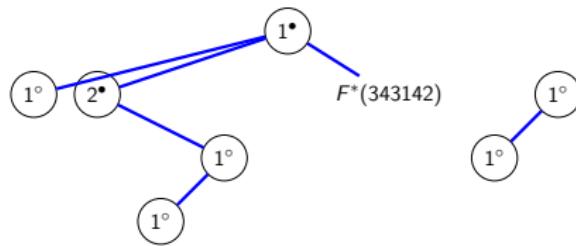
Right child

$$F^*(8967647523314) =$$



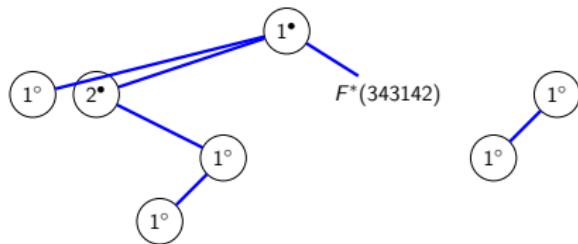
Blue forest of 8967647523314

$$F^*(8967647523314) =$$



Blue forest of 8967647523314

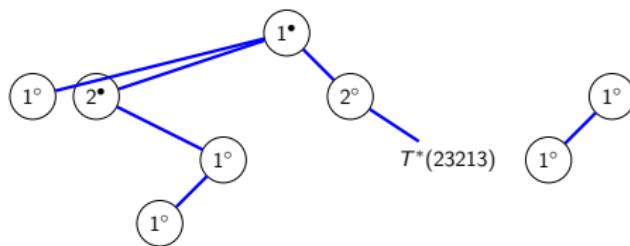
$$F^*(8967647523314) =$$



		•				•	
•			•				
							•
3	4	3	1	4	2		

Blue forest of 8967647523314

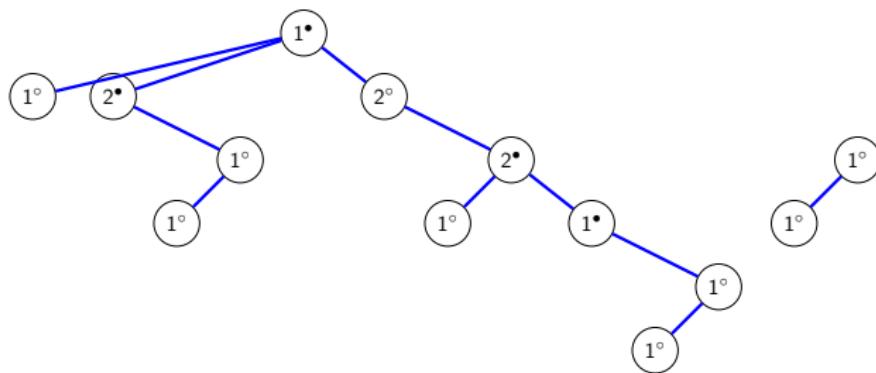
$$F^*(8967647523314) =$$



		•				•	
•			•				
X	←	←	←	←	←	○	
3	4	3	1	4	2		

Blue forest of 8967647523314

$$F^*(8967647523314) =$$



The basis \mathbb{O}

$$\mathbb{O}_{\bigcirc_1} := \mathbb{Q}_1,$$

$$\mathbb{O}_{t_1, \dots, t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$$\mathbb{O} \quad := \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$$

A diagram illustrating the construction of the basis \mathbb{O} . A central node labeled w is connected by several blue arrows to other nodes. One arrow points to a node labeled ℓ_1 , another to a node labeled ℓ_g , and others to intermediate nodes. This structure represents the free vector space generated by the \mathbb{O}_{ℓ_i} basis elements.

$$\mathbb{O}_{\bigcirc_{i^\alpha}} := \Psi_i^\alpha(\mathbb{O}_r).$$

A diagram illustrating the construction of the basis \mathbb{O}_{i^α} . A node labeled i^α is connected by a single blue arrow to a node labeled r . This structure represents the image of the basis element \mathbb{O}_r under the automorphism Ψ_i^α .

The basis \mathbb{O}

$$\mathbb{O}_{(1)} := \mathbb{Q}_1,$$

$$\mathbb{O}_{t_1, \dots, t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$$\mathbb{O} \quad := \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$$

$$\mathbb{O}_{i^\alpha} := \Psi_i^\alpha(\mathbb{O}_r).$$

Example

$$\mathbb{O} = \mathbb{Q}_{531442} + \mathbb{Q}_{521443} + \mathbb{Q}_{512443} - \mathbb{Q}_{534142} - \mathbb{Q}_{524143} - \mathbb{Q}_{514243} - \mathbb{Q}_{514432} - \mathbb{Q}_{524431} - \mathbb{Q}_{514423} + \mathbb{Q}_{541432} + \mathbb{Q}_{542431} + \mathbb{Q}_{541423}$$



The basis \mathbb{O}

$$\mathbb{O}_{\bigcirc_1} := \mathbb{Q}_1,$$

$$\mathbb{O}_{t_1, \dots, t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$$\mathbb{O} \quad := \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$$

$$\mathbb{O}_{i^\alpha} := \Psi_i^\alpha(\mathbb{O}_r).$$

Theorem [M.]

- $(\mathbb{O}_f)_{f \in \mathfrak{F}^*_n}$ is a basis of \mathbf{WQSym}_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{T}^*_n}$ is a basis of Prim_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{N}^*}$ is a basis of TPrim_n^* .



Theorems [M.]

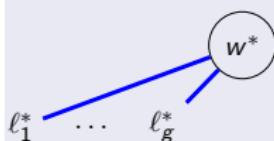
Theorem [M.]

- $(\mathbb{O}_f)_{f \in \mathfrak{F}_n^*}$ basis of \mathbf{WQSym}_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{T}_n^*}$ basis of Prim_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{P}_n^*}$ basis of TPrim_n^* .

Theorem [M.]

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_n}$ basis of \mathbf{WQSym}_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_n}$ basis of Prim_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{P}_n}$ basis of TPrim_n .

Rigidity

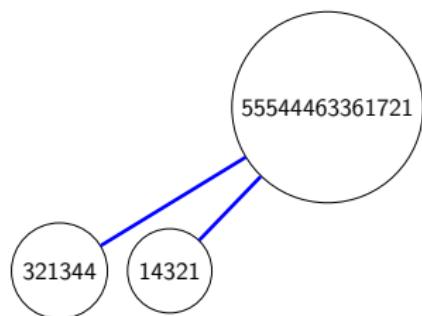


\forall bijection between red and blue irreducible words,
re-coloring of the skeletons



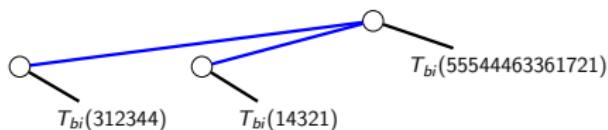
Bicolor forests through an example

$$T_{ske}^*(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, \\ 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9) =$$



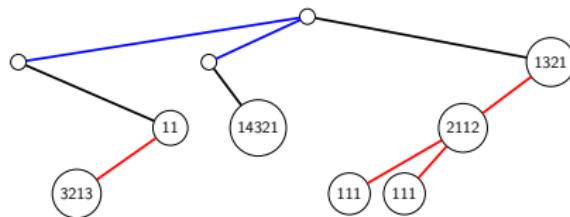
Bicolor forests through an example

$T_{bi}^*(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9,$
 $15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9)$



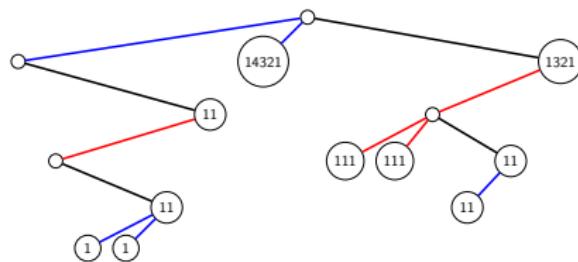
Bicolor forests through an example

$$T_{bi}^*(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, \\ 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9)$$



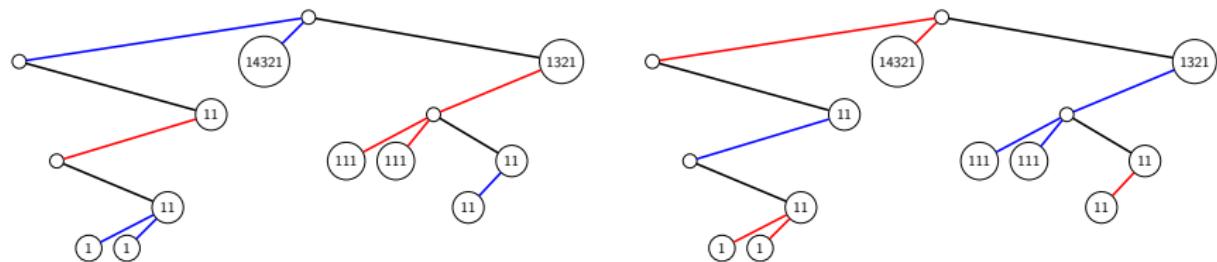
Bicolor forests through an example

$$T_{bi}^*(13, 13, 13, 12, 12, 12, 12, 14, 11, 11, 14, 9, \\15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9) =$$



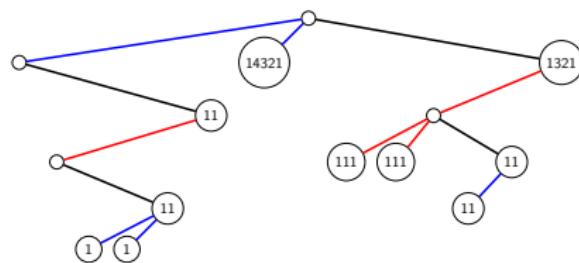
Bicolor forests through an example

$$T_{bi}^*(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, \\ 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9) =$$

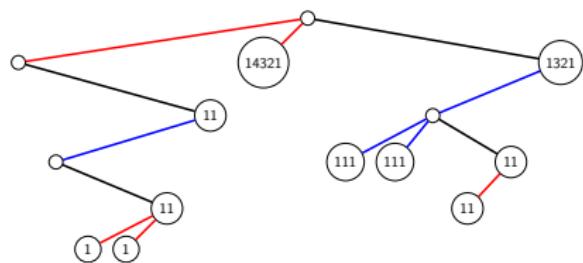


Bicolor forests through an example

$$T_{bi}^*(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, \\ 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9) =$$



$$T_{ske}(14, 12, 11, 13, 13, 14, 7, 10, 9, 8, 7, \\ 5, 15, 6, 3, 3, 4, 2, 2, 2, 1, 1, 1, 4, 5) =$$



Theorems [M.]

Theorem [M.]

- $(\mathbb{O}_f)_{f \in \mathfrak{F}_n^*}$ basis of \mathbf{WQSym}_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{T}_n^*}$ basis of Prim_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{P}_n^*}$ basis of TPrim_n^* .

Theorem [M.]

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_n}$ basis of \mathbf{WQSym}_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_n}$ basis of Prim_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{P}_n}$ basis of TPrim_n .

Bijection [M.]

Involution thanks to the bicolor forests.

Bidendriform isomorphism between \mathbf{WQSym} and \mathbf{WQSym}^* .