A bidendriform automorphism of WQSym

Seminar at York University

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Examples of Hopf algebras

- Planar binary trees, **PBT**, Loday-Ronco
- Non-commutative symmetric functions, **Sym**
- Quasi-symmetric functions, *QSym*
- Permutations, **FQSym**, Malvenuto-Reutenauer
- Packed words, **WQSym**, Hivert

Packed words

Definition

A word over the alphabet $\mathbb{N}_{>0}$ is packed if all the letters from 1 to its maximum m appears at least once.

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- ϵ

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- 123 132 213 231 312 321
  122 212 221 112 121 211 111

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Packed words of size 0, 1, 2 and 3

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- 123 132 213 231 312 321
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Packed words of size *n* [OEIS A000670]

n	1	2	3	4	5	6	7	8
$\mathbf{PW}_n$	1	3	13	75	541	4683	47293	545835

Paking

Example

$24154 \notin \mathbf{PW}$

Paking

Example

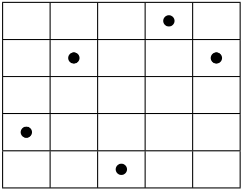
$24154 \notin \mathbf{PW}$ but $pack(24154) = 23143 \in \mathbf{PW}$

Paking

Example

$24154 \notin$ **PW** but $pack(24154) = 23143 \in$ **PW**

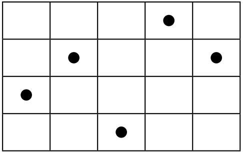
One representation : $\#\text{rows} \leq \#\text{columns}$



2 4 1 5 4

remove empty lines

→ pack →



2 3 1 4 3

Hopf algebra

Example

WQSym

- $3112 + 212 - 3\,212341 - \frac{5}{3}\,111$

Hopf algebra

Example

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$

Hopf algebra

Example

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
- $\mathbb{R}_{12}\mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$

Hopf algebra

Example

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
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- $\Delta(\mathbb{R}_{24231}) = \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_{1} + \mathbb{R}_{24231} \otimes \mathbb{R}_{\epsilon}$

Hopf algebra

Example

WQSym

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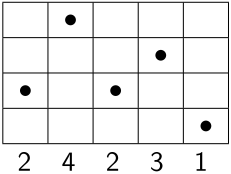
- unitary associative product $\cdot$
- counitary coassociative coproduct Δ
- Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

Deconcatenation (reduced)



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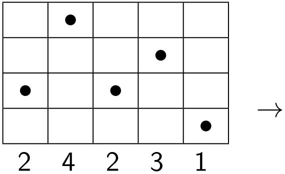

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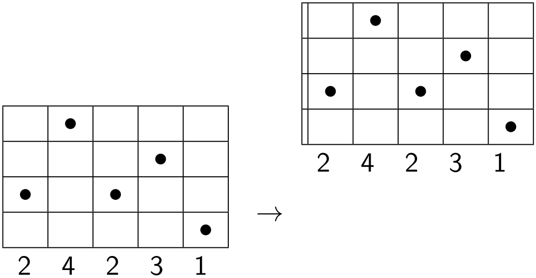

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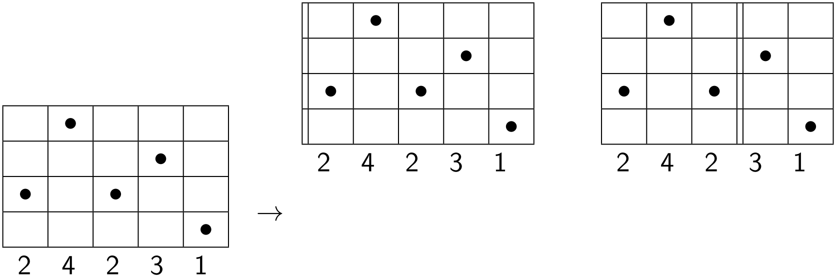

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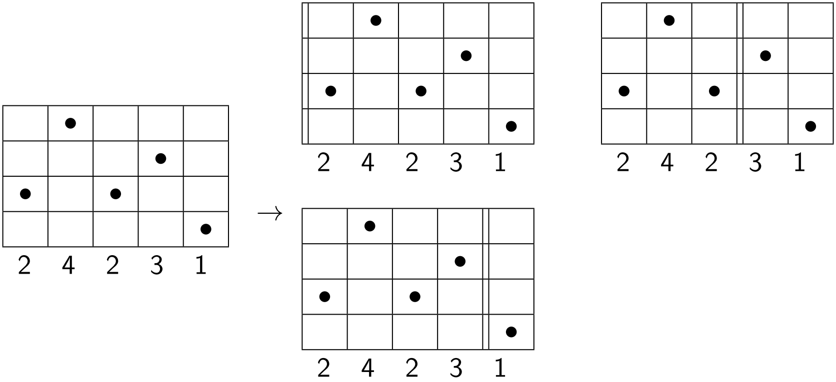

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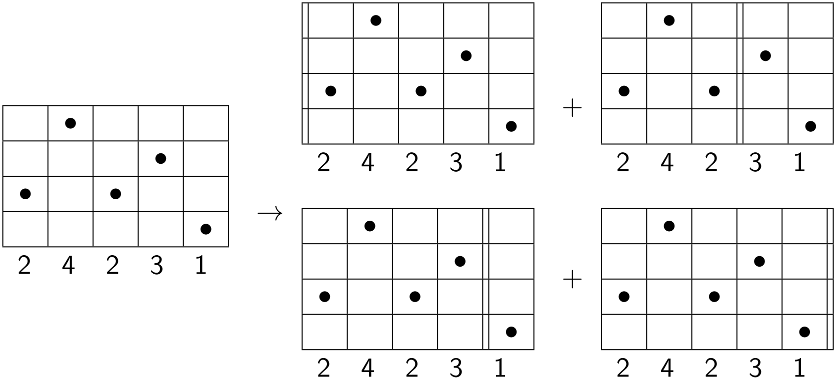
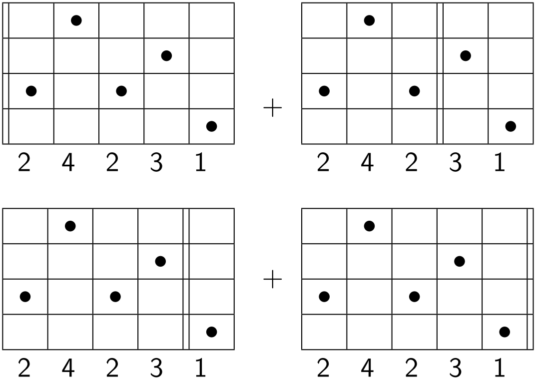


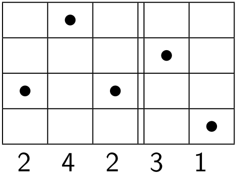
Deconcatenation (reduced)

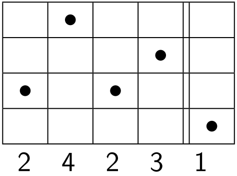
$\mathbb{R}_{24231} \rightarrow$



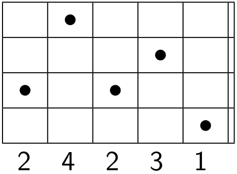
Deconcatenation (reduced)

$\mathbb{R}_{24231} \xrightarrow{\Delta}$ $\mathbb{R}_\epsilon \otimes \mathbb{R}_{24231}$ +



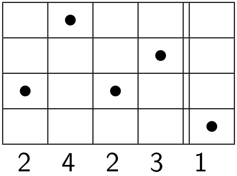


+



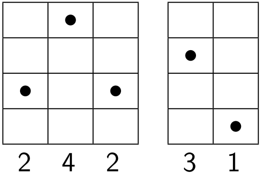
Deconcatenation (reduced)

$\mathbb{R}_{24231}$ $\overset{\Delta}{\to}$

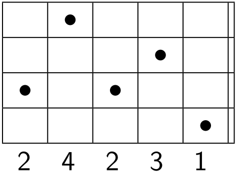


$+ \; \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231}$

$+$

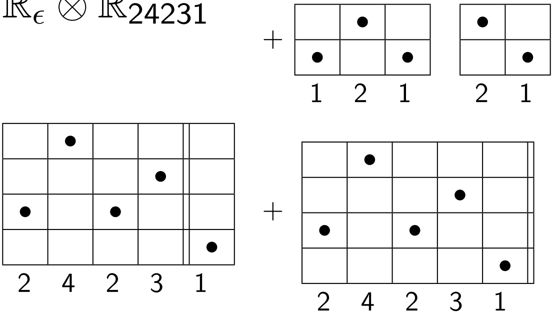


$+$



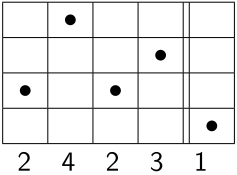
Deconcatenation (reduced)

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231}$$

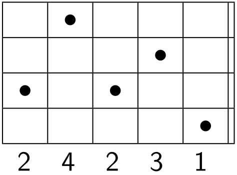


Deconcatenation (reduced)

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21}$$



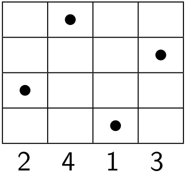
+



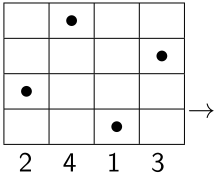
Deconcatenation (reduced)

$$\mathbb{R}_{24231} \xrightarrow{\Delta} \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_{1} + \mathbb{R}_{24231} \otimes \mathbb{R}_{\epsilon}$$

Horizontal disassembly



Horizontal disassembly



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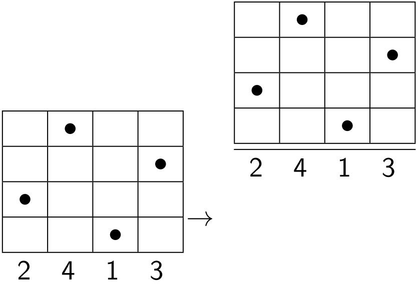

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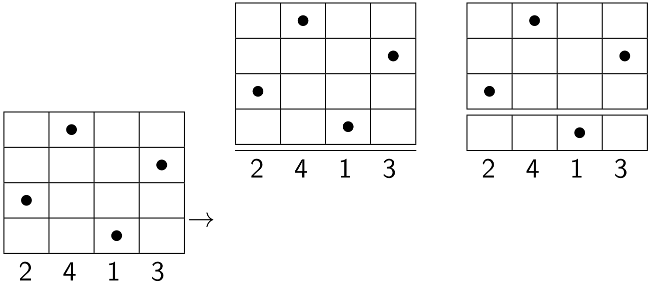

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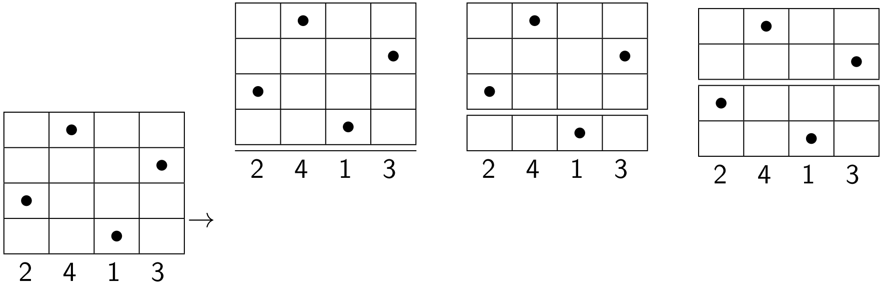
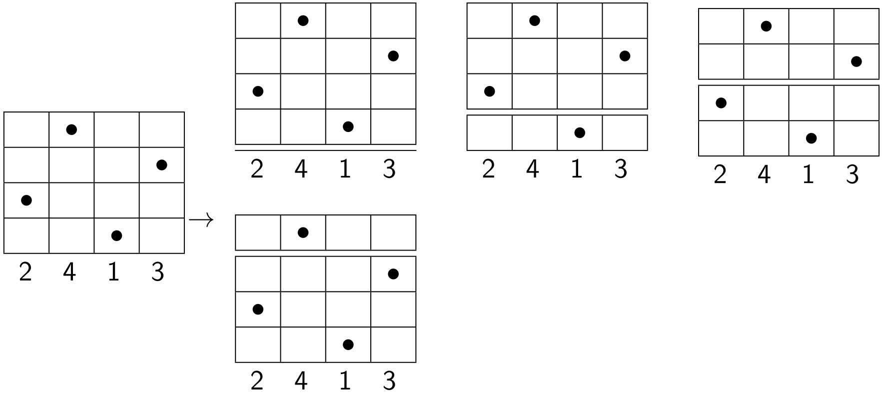


Horizontal disassembly



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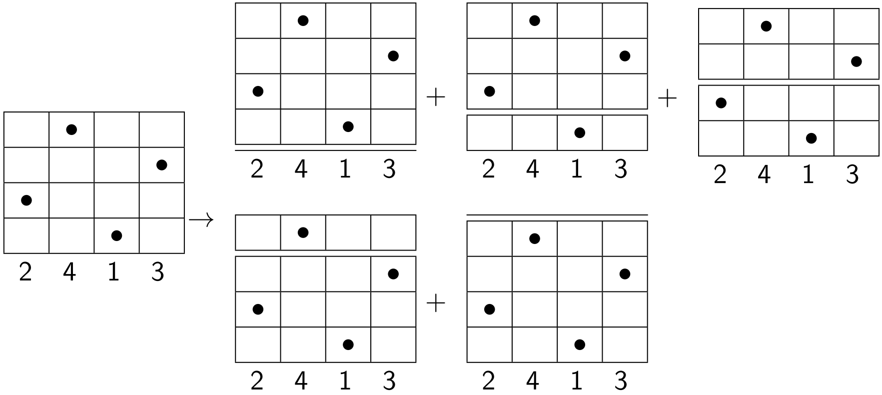
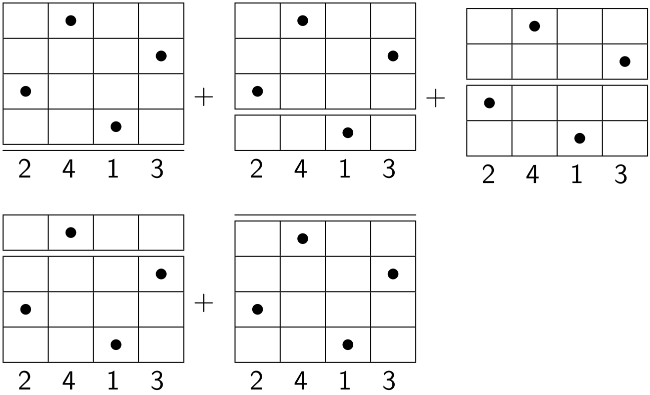
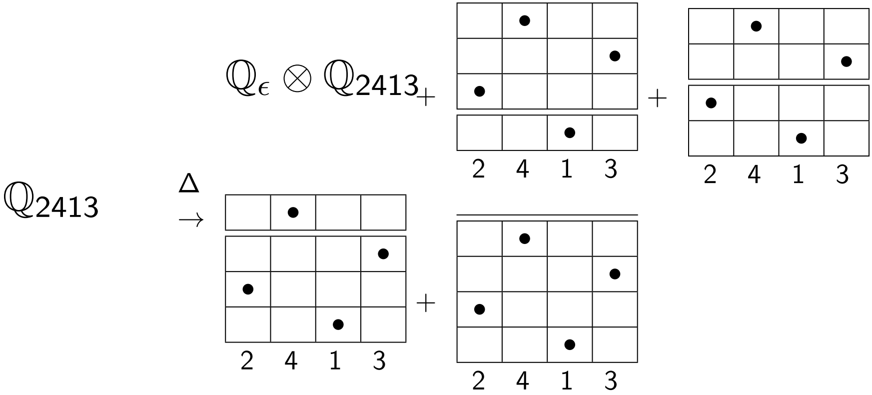


Horizontal disassembly

$\mathbb{Q}_{2413} \xrightarrow{\Delta}$



Horizontal disassembly



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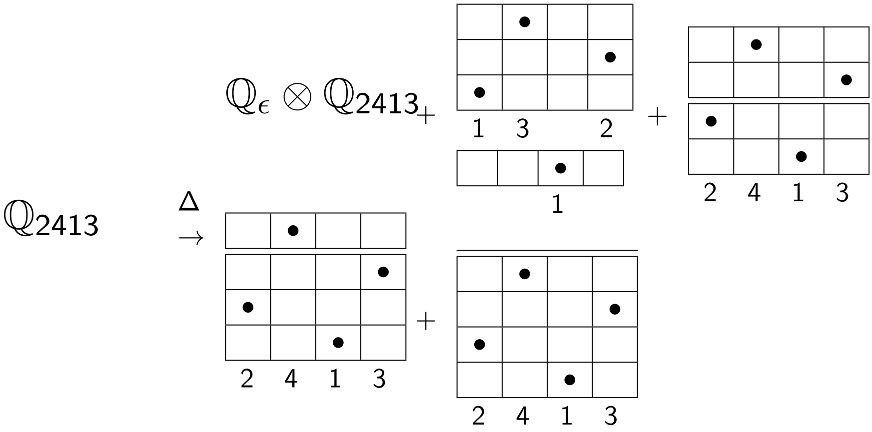
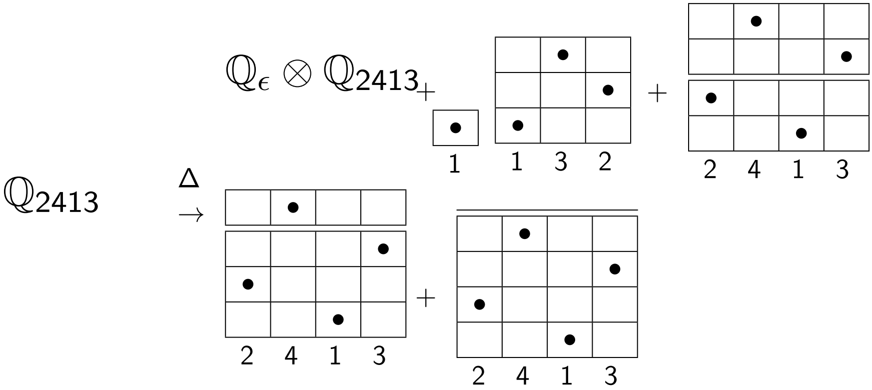
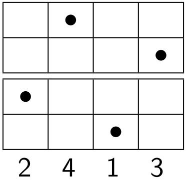


Horizontal disassembly

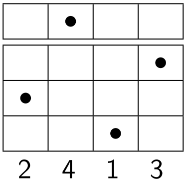


Horizontal disassembly

$$\mathbb{Q}_\epsilon \otimes \mathbb{Q}_{2413} + \mathbb{Q}_1 \otimes \mathbb{Q}_{132} +$$



$\mathbb{Q}_{2413} \xrightarrow{\Delta}$



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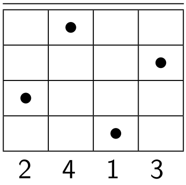

Horizontal disassembly

$$\mathbb{Q}_{2413} \xrightarrow{\Delta} \mathbb{Q}_\epsilon \otimes \mathbb{Q}_{2413} + \mathbb{Q}_1 \otimes \mathbb{Q}_{132} + \mathbb{Q}_{21} \otimes \mathbb{Q}_{21} + \mathbb{Q}_{213} \otimes \mathbb{Q}_1 + \mathbb{Q}_{2413} \otimes \mathbb{Q}_\epsilon$$

Auto-duality

- $\mathbb{R}$ and $\mathbb{Q}$ bases of **WQSym** and **WQSym***

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Auto-duality

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- 2005 Foissy demonstrates the self-duality of bidendriform bialgebra (rigidity)
- No explicit isomorphism

Half coproducts

Example of left and right coproducts

- $\tilde{\Delta}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321} + \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_{1}$

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- $\Delta_{\prec}(\mathbb{R}_{2425531}) = \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_{1}$
- $\Delta_{\succ}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321}$

Half coproducts

Definitions

- $\Delta_{\prec}(\mathbb{R}_u) := \sum_{\substack{i=k \\ \{u_1,\dots,u_i\}\cap\{u_{i+1},\dots,u_n\}=\emptyset \\ u_k=\max(u)}}^{n-1} \mathbb{R}_{pack(u_1\cdots u_i)} \otimes \mathbb{R}_{pack(u_{i+1}\cdots u_n)}$,
- $\Delta_{\succ}(\mathbb{R}_u) := \sum_{\substack{i=1 \\ \{u_1,\dots,u_i\}\cap\{u_{i+1},\dots,u_n\}=\emptyset \\ u_k=\max(u)}}^{k-1} \mathbb{R}_{pack(u_1\cdots u_i)} \otimes \mathbb{R}_{pack(u_{i+1}\cdots u_n)}$

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Bidendriform bialgebra

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- Refinement of associativity and co-associativity
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Theorem [Foissy]

If A is a bidendriform bialgebra then A is freely generated by $\mathrm{TPrim}(A)$ as a dendriform algebra.

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If A is a bidendriform bialgebra then A is freely generated by $\mathrm{TPrim}(A)$ as a dendriform algebra.

Series

n	1	2	3	4	5	6	7	8
$\mathbf{WQSym}_n$	1	3	13	75	541	4 683	47 293	545 835
TPrim_n	1	1	4	28	240	2 384	26 832	337 168

Bidendriform bialgebra

Definition

- Refinement of associativity and co-associativity
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- Refinement of the Hopf relation
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Theorem [Foissy]

If A is a bidendriform bialgebra then A is freely generated by $\mathrm{TPrim}(A)$ as a dendriform algebra.

Corollary

WQSym is self-dual.

Definitions

Primitive element

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

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Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

Definitions

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Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

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$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

Totally primitive element

P is a totally primitive element $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

Definitions

Primitive element

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

$\tilde{\Delta}(\mathbb{R}_{1213}) = \Delta_{\succ}(\mathbb{R}_{1213}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

$\tilde{\Delta}(\mathbb{R}_{2321}) = \Delta_{\prec}(\mathbb{R}_{2321}) = \mathbb{R}_{121} \otimes \mathbb{R}_{1}$

Totally primitive element

P is a totally primitive element $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

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$\tilde{\Delta}(\mathbb{R}_{12443}) = \mathbb{R}_{1233} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{12} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{1332}$

$\tilde{\Delta}(\mathbb{R}_{21443}) = \mathbb{R}_{2133} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{21} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{1332}$

$\tilde{\Delta}(\mathbb{R}_{23441}) = \mathbb{R}_{1233} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{12} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{2331}$

$\tilde{\Delta}(\mathbb{R}_{32441}) = \mathbb{R}_{2133} \otimes \mathbb{R}_{1} \qquad \mathbb{R}_{21} \otimes \mathbb{R}_{221} + \mathbb{R}_{1} \otimes \mathbb{R}_{2331}$

My goal

Explicit bidendriform isomorphism between **WQSym** and it's dual

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$\Updownarrow$

Explicit isomorphism between TPrim(**WQSym**) and it's dual

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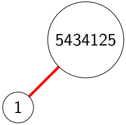
$\Updownarrow$

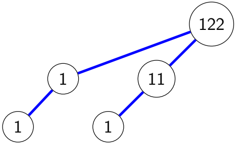
Explicit isomorphism between TPrim(**WQSym**) and it's dual

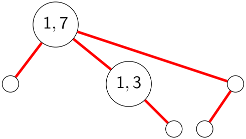
Construction of two bases of totally primitive
(in **WQSym** and **WQSym***)

Biplane Forests, representation of decompositions

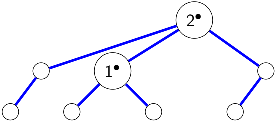


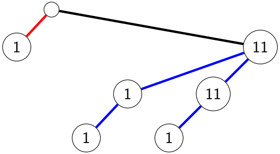
Skeletons



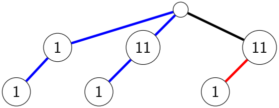


Paked trees



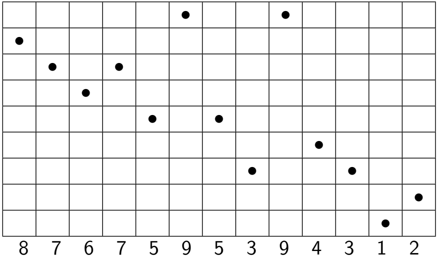


Bicolors trees



Red skeleton of 8767595394312

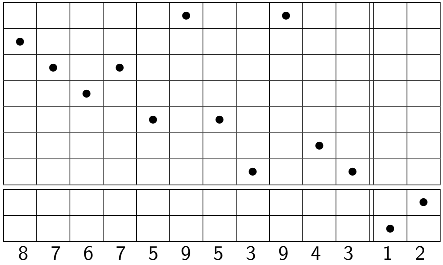
$F_{ske}(8767595394312)$



Red skeleton of 8767595394312

$F_{ske}(8767595394312)$

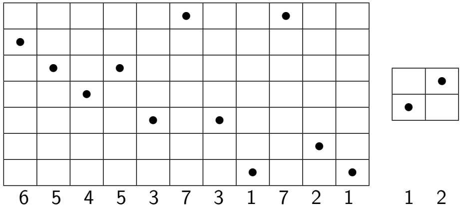
Global descents factorisation



Red skeleton of 8767595394312

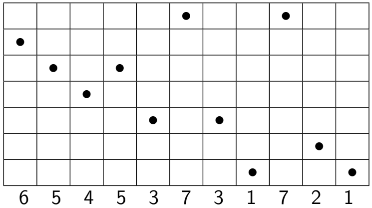
$F_{ske}(8767595394312) = T_{ske}(65453731721)\, T_{ske}(12)$

Global descents factorisation + paking



Red skeleton of 8767595394312

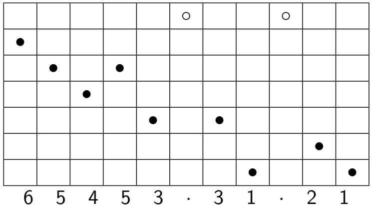
$F_{ske}(8767595394312) =$
$T_{ske}(65453731721)\,T_{ske}(12)$



Red skeleton of 8767595394312

$F_{ske}(8767595394312) = T_{ske}(65453731721)\,T_{ske}(12)$

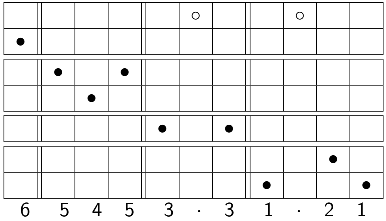
Remove all the occurences of the maximal value



Red skeleton of 8767595394312

$F_{ske}(8767595394312) = T_{ske}(65453731721)\,T_{ske}(12)$

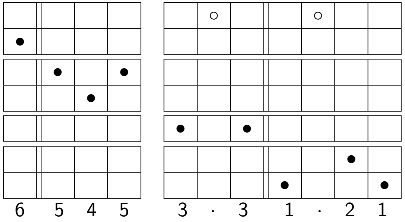
Global descents factorisation



Red skeleton of 8767595394312

$F_{ske}(8767595394312) =$
$T_{ske}(65453731721)\,T_{ske}(12)$

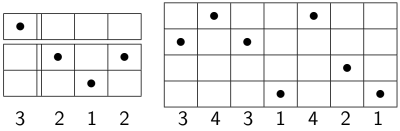
Distinction of two groups of factors



Red skeleton of 8767595394312

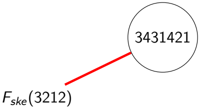
$F_{ske}(8767595394312) = T_{ske}(65453731721)\,T_{ske}(12)$

Reinsert the removed letters + paking

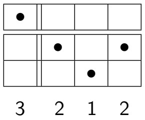


Red skeleton of 8767595394312

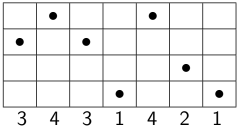
$F_{ske}(8767595394312) =$
$T_{ske}(65453731721)\, T_{ske}(12) =$



$T_{ske}(12)$



3431421 is
Red irreductible



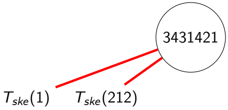
Red irreductible

A packed word w is **red irreductible** if it is not decomposable by this algorithm.

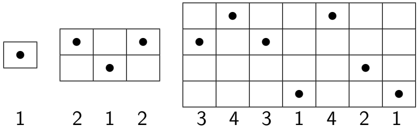
Red skeleton of 8767595394312

$F_{ske}(8767595394312) =$
$T_{ske}(65453731721)\,T_{ske}(12) =$

Loop



$T_{ske}(12)$

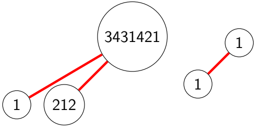


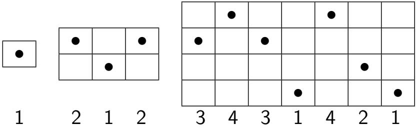
Red irreductible

A packed word w is **red irreductible** if it is not decomposable by this algorithm.

Red skeleton of 8767595394312

$F_{ske}(8767595394312) =$





Red irreductible

A packed word w is **red irreductible** if it is not decomposable by this algorithm.

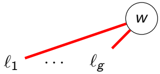
$\forall n, \#RedIrreductible_n = \dim(\mathrm{TPrim}_n).$

First part for basis $\mathbb{P}$

$$\mathbb{P}_{\textcircled{1}} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

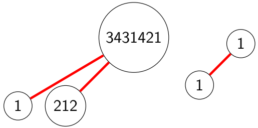
$\mathbb{P}$



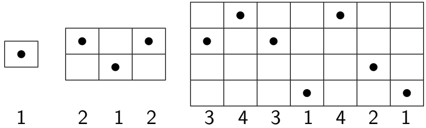
$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle.$$

Red forest of 8767595394312

$F(8767595394312) =$

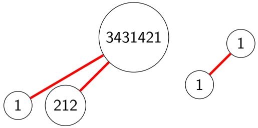


The right part!

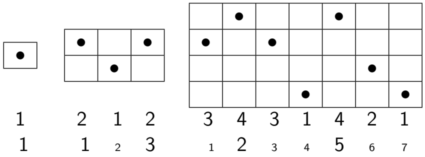


Red forest of 8767595394312

$F(8767595394312) =$



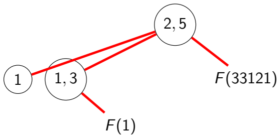
Positions of max

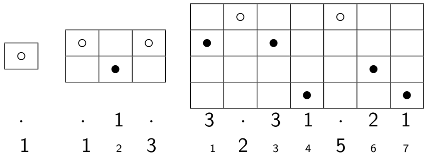


Red forest of 8767595394312

Right children

$F(8767595394312) =$

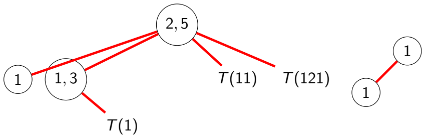


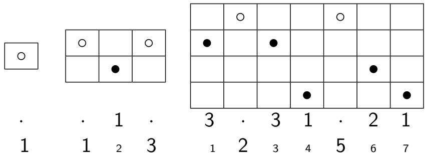


Red forest of 8767595394312

Loop again

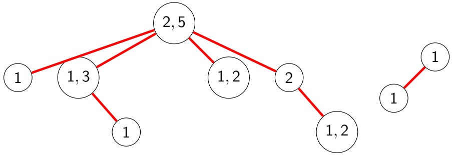
$F(8767595394312) =$





Red forest of 8767595394312

$F(8767595394312) =$

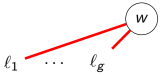


The basis $\mathbb{P}$

$$\mathbb{P}_{①} := \mathbb{R}_1,$$

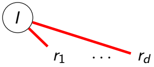
$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$



$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$

$\mathbb{P}$



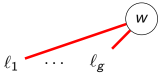
$$:= \Phi_I(\mathbb{P}_{r_1,\dots,r_d}).$$

The basis ℙ

$$\mathbb{P}_{\textcircled{1}} := \mathbb{R}_1,$$

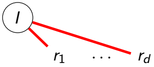
$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$



$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$

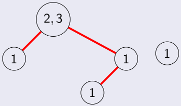
$\mathbb{P}$



$$:= \Phi_I(\mathbb{P}_{r_1,\dots,r_d}).$$

Example

$\mathbb{P}$

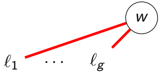


$$= \mathbb{R}_{235541} - \mathbb{R}_{245531} - \mathbb{R}_{244531} - \mathbb{R}_{245431} - \mathbb{R}_{254431} + \mathbb{R}_{325541} - \mathbb{R}_{425531} - \mathbb{R}_{524431} + \mathbb{R}_{352541} - \mathbb{R}_{452531} + \mathbb{R}_{355241} - \mathbb{R}_{455231} + \mathbb{R}_{344521} + \mathbb{R}_{345421} + \mathbb{R}_{354421} + \mathbb{R}_{534421}$$

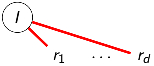
The basis $\mathbb{P}$

$$\mathbb{P}_{\textcircled{1}} := \mathbb{R}_1,$$

$$\mathbb{P}_{t_1,\ldots,t_k} := (\ldots(\mathbb{P}_{t_k} \prec \ldots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$ 

$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \ldots, \mathbb{P}_{\ell_g}; \mathbb{P}_{T(w)} \rangle,$$

$\mathbb{P}$ 

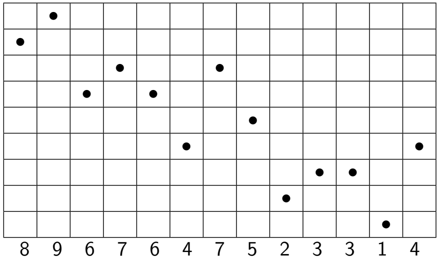
$$:= \Phi_I(\mathbb{P}_{r_1,\ldots,r_d}).$$

Theorem [M.]

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_n}$ is a basis of **WQSym**$_n$,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_n}$ is a basis of Prim$_n$,
- $(\mathbb{P}_t)_{t \in \mathfrak{P}_n}$ is a basis of TPrim$_n$.

Blue skeleton of 8967647523314

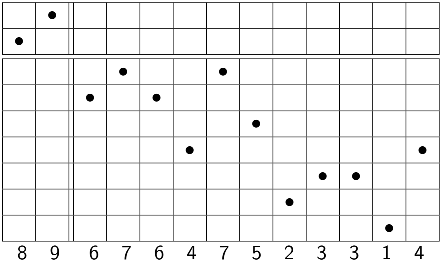
$F^*_{ske}(8967647523314)$



Blue skeleton of 8967647523314

$F^*_{ske}(8967647523314)$

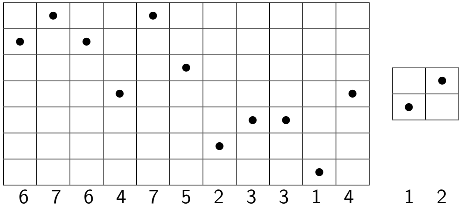
Global descents factorisation



Blue skeleton of 8967647523314

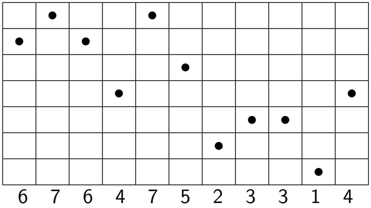
$F^{*}_{ske}(8967647523314) = T^{*}_{ske}(67647523314)\,T^{*}_{ske}(12)$

Global descents factorisation + paking + swap



Blue skeleton of 8967647523314

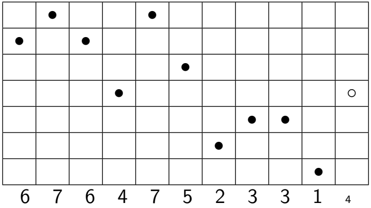
$F^*_{ske}(8967647523314) =$
$T^*_{ske}(67647523314)\, T^*_{ske}(12)$



Blue skeleton of 8967647523314

$F^*_{ske}(8967647523314) = T^*_{ske}(67647523314)\,T^*_{ske}(12)$

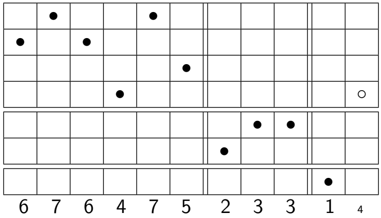
Remove of the last letter



Blue skeleton of 8967647523314

$F^*_{ske}(8967647523314) =$
$T^*_{ske}(67647523314)\, T^*_{ske}(12)$

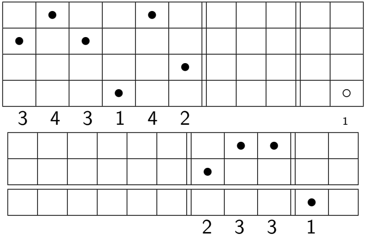
Global descents factorisation



Blue skeleton of 8967647523314

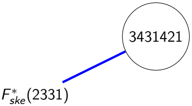
$F^*_{ske}(8967647523314) =$
$T^*_{ske}(67647523314)\,T^*_{ske}(12)$

Distinction of two groups of factors

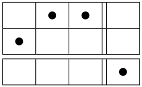


Blue skeleton of 8967647523314

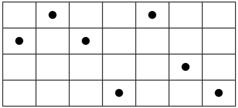
$F^*_{ske}(8967647523314) =$
$T^*_{ske}(67647523314)\, T^*_{ske}(12) =$



$T^*_{ske}(12)$



3431421 is
Blue irreductible



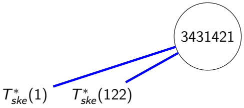
Blue irreductible

A packed word w is **blue irreductible** if it is not decomposable by this algorithm.

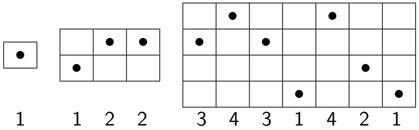
Blue skeleton of 8967647523314

$F^*_{ske}(8967647523314) =$ Loop

$T^*_{ske}(67647523314)\,T^*_{ske}(12) =$



$T^*_{ske}(12)$

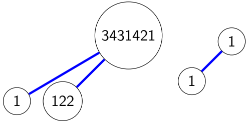


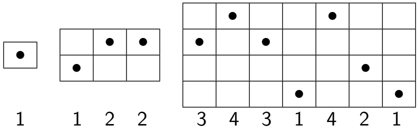
Blue irreductible

A packed word w is **blue irreductible** if it is not decomposable by this algorithm.

Blue skeleton of 8967647523314

$F^*_{ske}(8967647523314) =$





Blue irreductible

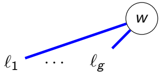
A packed word w is **blue irreductible** if it is not decomposable by this algorithm.

$$\forall n, \#BlueIrreductible_n = \#RedIrreductible_n = \dim(\mathrm{TPrim}_n).$$

First part for basis $\mathbb{O}$

$$\mathbb{O}_{\textcircled{1}} := \mathbb{Q}_1,$$

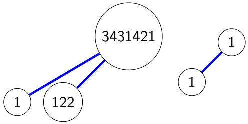
$$\mathbb{O}_{t_1,\dots,t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$\mathbb{O}$ 

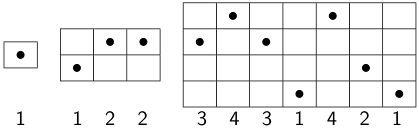
$$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle.$$

Blue forest of 8967647523314

$F^*(8967647523314) =$

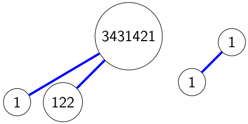


The right part!

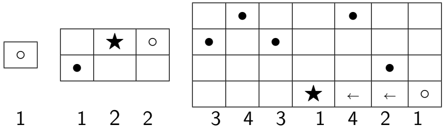


Blue forest of 8967647523314

$F^*(8967647523314) =$



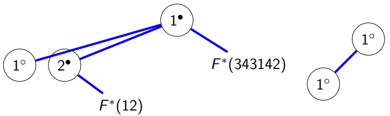
The last lettre appears in the rest of the word?

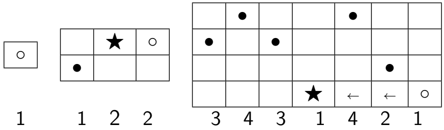


Blue forest of 8967647523314

Right child

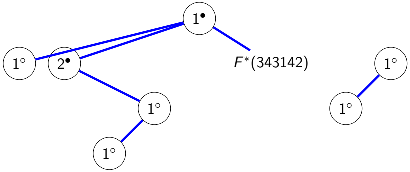
$F^*(8967647523314) =$





Blue forest of 8967647523314

$F^*(8967647523314) =$



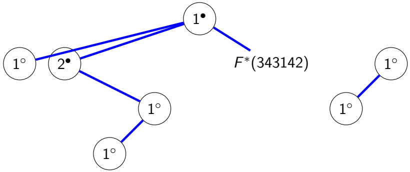
○
1

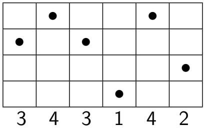
	★	○
●		
1	2	2

	●			●		
●		●				
					●	
			★	←	←	○
3	4	3	1	4	2	1

Blue forest of 8967647523314

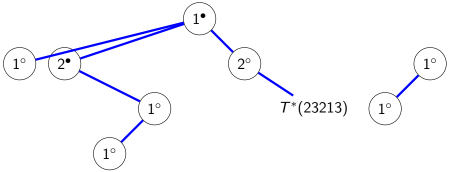
$F^*(8967647523314) =$





Blue forest of 8967647523314

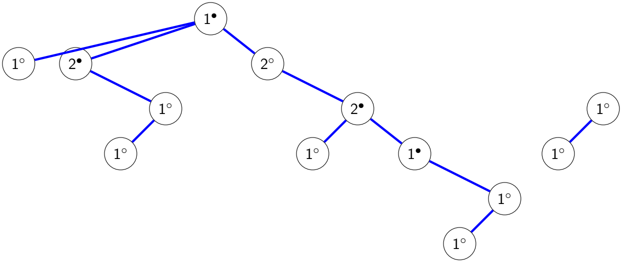
$F^*(8967647523314) =$



	●			●	
●		●			
×	←	←	←	←	○
			●		
3	4	3	1	4	2

Blue forest of 8967647523314

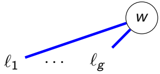
$F^*(8967647523314) =$



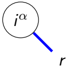
The basis 𝕆

$$\mathbb{O}_{\textcircled{1}} := \mathbb{Q}_1,$$

$$\mathbb{O}_{t_1,\ldots,t_k} := (\ldots(\mathbb{O}_{t_k} \prec \ldots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$\mathbb{O}$ 

$$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \ldots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$$

$\mathbb{O}$ 

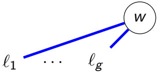
$$:= \Psi_i^\alpha(\mathbb{O}_r).$$

The basis $\mathbb{O}$

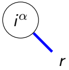
$\mathbb{O}$ 

$:= \mathbb{Q}_1,$

$$\mathbb{O}_{t_1,\ldots,t_k} := (\ldots(\mathbb{O}_{t_k} \prec \ldots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

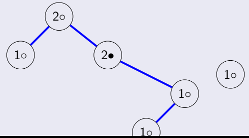
$\mathbb{O}$ 

$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \ldots, \mathbb{O}_{\ell_g} ; \mathbb{O}_{T^*(w)} \rangle,$

$\mathbb{O}$ 

$:= \Psi_i^\alpha(\mathbb{O}_r).$

Example

$\mathbb{O}$ 

$=$

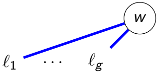
$$\mathbb{Q}_{531442} + \mathbb{Q}_{521443} + \mathbb{Q}_{512443} - \mathbb{Q}_{534142} - \mathbb{Q}_{524143} - \mathbb{Q}_{514243} - \mathbb{Q}_{514432} - \mathbb{Q}_{524431} - \mathbb{Q}_{514423} + \mathbb{Q}_{541432} + \mathbb{Q}_{542431} + \mathbb{Q}_{541423}$$

The basis $\mathbb{O}$

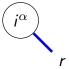
$\mathbb{O}$ 

$:= \mathbb{Q}_1,$

$$\mathbb{O}_{t_1,\dots,t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$\mathbb{O}$ 

$:= \langle \mathbb{O}_{\ell_1}, \mathbb{O}_{\ell_2}, \dots, \mathbb{O}_{\ell_g}; \mathbb{O}_{T^*(w)} \rangle,$

$\mathbb{O}$ 

$:= \Psi_i^\alpha(\mathbb{O}_r).$

Theorem [M.]

- $(\mathbb{O}_f)_{f\in\mathfrak{F}^*{}_n}$ is a basis of $\mathbf{WQSym}_n^*$,
- $(\mathbb{O}_t)_{t\in\mathfrak{T}^*{}_n}$ is a basis of Prim_n^*,
- $(\mathbb{O}_t)_{t\in\mathfrak{B}^*{}_n}$ is a basis of TPrim_n^*.

Theorems [M.]

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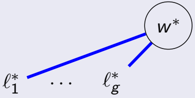
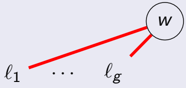
Rigidity

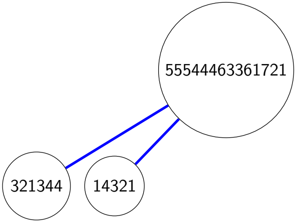


$\forall$ bijection between red and blue irreducible words, re-coloring of the skeletons



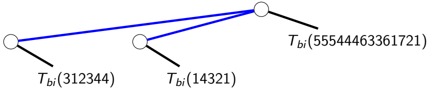
Bicolor forests through an example

$$T^*_{ske}(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9) =$$



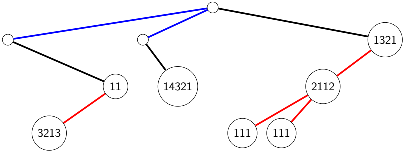
Bicolor forests through an example

$T^*_{bi}(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9,$
$15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9)$



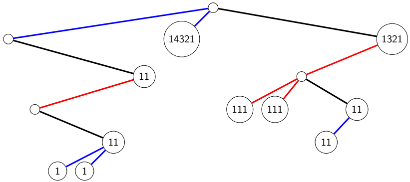
Bicolor forests through an example

$T^*_{bi}(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9)$



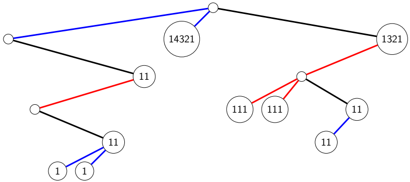
Bicolor forests through an example

$$T^*_{bi}(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9) =$$



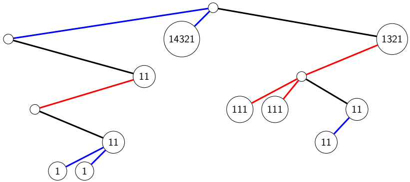
Bicolor forests through an example

$$T^*_{bi}(13, 13, 13, 12, 12, 12, 14, 11, 11, 14, 9, 15, 10, 5, 8, 7, 6, 5, 3, 2, 1, 3, 4, 4, 9) =$$

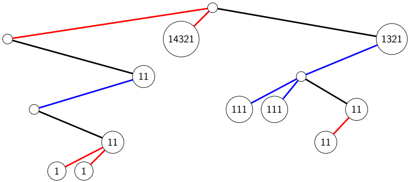


Bicolor forests through an example

$$T^*_{bi}(13,13,13,12,12,12,14,11,11,14,9,15,10,5,8,7,6,5,3,2,1,3,4,4,9) =$$



$$T_{ske}(14,12,11,13,13,14,7,10,9,8,7,5,15,6,3,3,4,2,2,2,1,1,1,4,5) =$$



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Bijection [M.]

Involution thanks to the bicolor forests.
Bidendriform isomorphism between **WQSym** and $\mathbf{WQSym}^*$.