

Un auto-morphisme bidendriforme de WQSym

Séminaire IRIF

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Directeurs:

Florent Hivert

Viviane Pons

18 Juin 2020

Introduction

- $abcd + badc - 3bcad - \frac{5}{3}dcba$

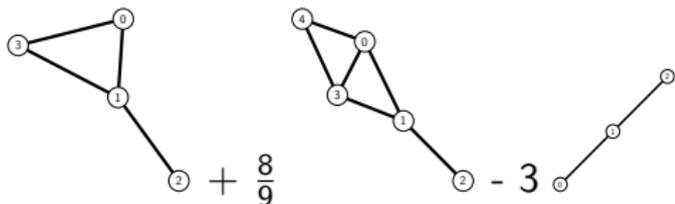
Introduction

$$\bullet \text{abcd} + \text{badc} - 3\text{bcad} - \frac{5}{3}\text{dcba}$$

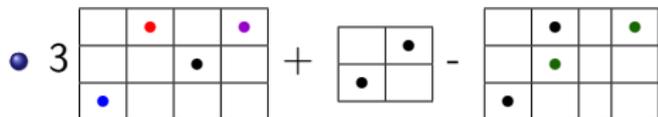
$$\bullet \text{Diagram 1} + \frac{8}{9} \text{Diagram 2} - 3 \text{Diagram 3}$$

Introduction

$$\bullet \text{abcd} + \text{badc} - 3\text{bcad} - \frac{5}{3}\text{dcba}$$



$$\bullet + \frac{8}{9}$$



$$\bullet 3$$

Exemples d'algèbres de Hopf

- Arbres binaires, **PBT**, Loday-Ronco
- Fonctions symétriques non-commutatives, **Sym**
- Fonctions quasi-symétriques, *QSym*
- Permutations, **FQSym**, Malvenuto-Reutenauer
- Mots tassés, **WQSym**, Hivert

Mots tassés

Définition

Un mot sur l'alphabet $\mathbb{N}_{>0}$ est dit **tassé** si toutes les lettres de 1 à son maximum m apparaissent au moins une fois.

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Mots tassés de tailles 0, 1, 2 et 3

- ϵ

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- 1
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Mots tassés de tailles 0, 1, 2 et 3

- ϵ
- 1
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- 123 132 213 231 312 321
122 212 221 112 121 211 111

Mots tassés

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Mots tassés de tailles 0, 1, 2 et 3

- ϵ
- 1
- 12 21 11
- 123 132 213 231 312 321
122 212 221 112 121 211 111

Série des mots tassés

n	1	2	3	4	5	6	7	8
PW_n	1	3	13	75	541	4683	47293	545835

Tassement

Exemple

24154 \notin **PW**

Tassement

Exemple

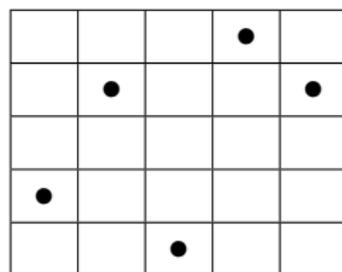
$24154 \notin \mathbf{PW}$ mais $pack(24154) = 23143 \in \mathbf{PW}$

Tassement

Exemple

$24154 \notin \mathbf{PW}$ mais $pack(24154) = 23143 \in \mathbf{PW}$

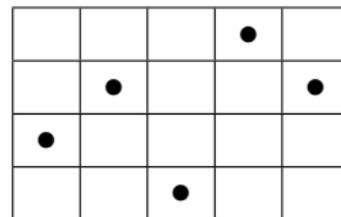
Une représentation : $\#lignes \leq \#colonnes$



2 4 1 5 4

retrait lignes vides

→ pack →



2 3 1 4 3

Algèbre de Hopf

Exemple

WQSym

- $3112 + 212 - 3 \cdot 212341 - \frac{5}{3} \cdot 111$

Algèbre de Hopf

Exemple

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$

Algèbre de Hopf

Exemple

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
- $\mathbb{R}_{12}\mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$

Algèbre de Hopf

Exemple

WQSym

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- $\Delta(\mathbb{R}_{24231}) = \mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_1 + \mathbb{R}_{24231} \otimes \mathbb{R}_\epsilon$

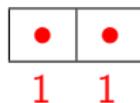
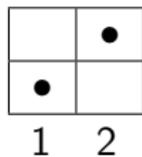
Algèbre de Hopf

Exemple

WQSym

- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
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 - $\Delta(\mathbb{R}_{24231}) = \mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_1 + \mathbb{R}_{24231} \otimes \mathbb{R}_\epsilon$
- Un produit associatif unitaire \cdot
 - Un coproduit coassociatif counitaire Δ
 - La relation de Hopf $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

Produit de mélange sur les mots tassés



Produit de mélange sur les mots tassés

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \boxtimes \quad \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} =$$

1 2 1 1

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & \bullet & & \\ \hline \bullet & & & \\ \hline \end{array}$$

1 2 3 3

Produit de mélange sur les mots tassés

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \sqcup \quad \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} =$$

1 2 1 1

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array}$$

1 2 3 3 1 3 2 3 1 3 3 2

$$\begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & \bullet & \\ \hline & \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array}$$

3 1 2 3 3 1 3 2 3 3 1 2

Produit de mélange sur les mots tassés

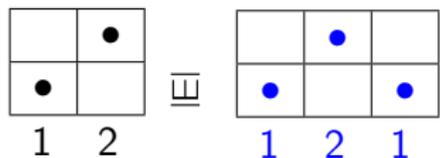
$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \begin{array}{c} 1 \\ 2 \end{array} \sqcup \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{c} 1 \\ 1 \end{array} = \mathbb{R}$$

$$\mathbb{R}_{12}\mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$$

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 3 \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & & \bullet \\ \hline & & \bullet & \\ \hline \bullet & & & \\ \hline \end{array} \begin{array}{c} 1 \\ 3 \\ 2 \\ 3 \end{array} + \begin{array}{|c|c|c|c|} \hline & \bullet & \bullet & \\ \hline & & & \bullet \\ \hline \bullet & & & \\ \hline \end{array} \begin{array}{c} 1 \\ 3 \\ 3 \\ 2 \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline \bullet & & & \bullet \\ \hline & & \bullet & \\ \hline & \bullet & & \\ \hline \end{array} \begin{array}{c} 3 \\ 1 \\ 2 \\ 3 \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & & \bullet & \\ \hline & & & \bullet \\ \hline & \bullet & & \\ \hline \end{array} \begin{array}{c} 3 \\ 1 \\ 3 \\ 2 \end{array} + \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & & \\ \hline & & & \bullet \\ \hline & & \bullet & \\ \hline \end{array} \begin{array}{c} 3 \\ 3 \\ 1 \\ 2 \end{array}$$

Produit de mélange sur les valeurs



Produit de mélange sur les valeurs

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \underline{\underline{\boxplus}} \quad \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \bullet \\ \hline \end{array} =$$

1 2
1 2 1

			•	
		•		•
	•			
•				

1
2
3
4
3

Produit de mélange sur les valeurs

$$\begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array} \quad \underline{\boxplus} \quad \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \bullet \\ \hline \end{array} =$$

1 2
1 2 1

$$\begin{array}{|c|c|c|c|c|} \hline & & & \bullet & \\ \hline & & & & \bullet \\ \hline & & \bullet & & \\ \hline \bullet & & & & \\ \hline \end{array} +$$

1 2 3 4 3

$$\begin{array}{|c|c|c|c|c|} \hline & & & \bullet & \\ \hline & \bullet & & & \\ \hline & & \bullet & & \bullet \\ \hline \bullet & & & & \\ \hline \end{array} +$$

1 3 2 4 2

$$\begin{array}{|c|c|c|c|c|} \hline & \bullet & & & \\ \hline & & & \bullet & \\ \hline & & \bullet & & \bullet \\ \hline \bullet & & & & \\ \hline \end{array}$$

1 4 2 3 2

$$\begin{array}{|c|c|c|c|c|} \hline & & & \bullet & \\ \hline & & \bullet & & \\ \hline \bullet & & & & \\ \hline & & \bullet & & \bullet \\ \hline \end{array} +$$

2 3 1 4 1

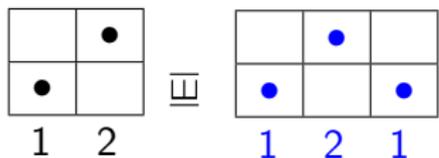
$$\begin{array}{|c|c|c|c|c|} \hline & \bullet & & & \\ \hline & & & \bullet & \\ \hline \bullet & & & & \\ \hline & & \bullet & & \bullet \\ \hline \end{array} +$$

2 4 1 3 1

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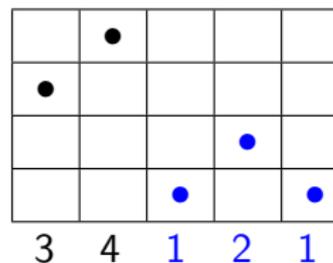
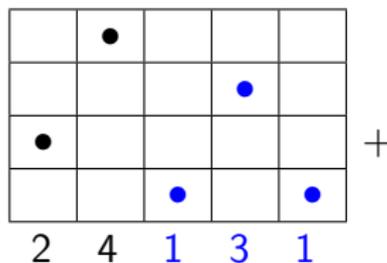
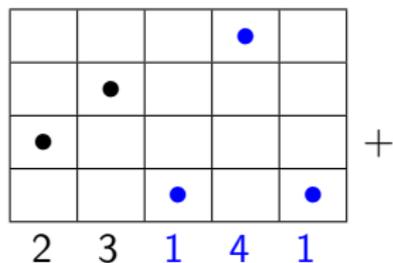
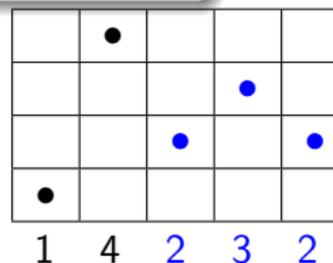
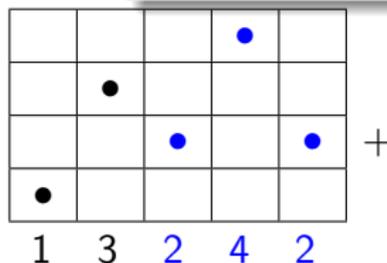
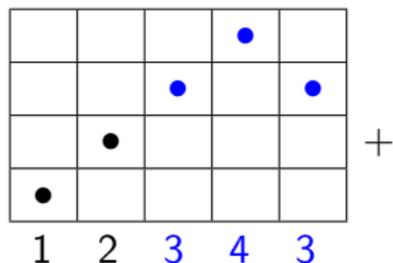
3 4 1 2 1

Produit de mélange sur les valeurs

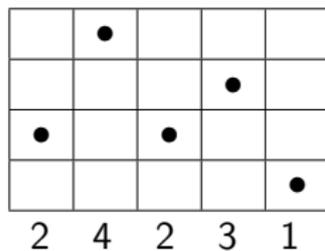


\mathbb{Q}

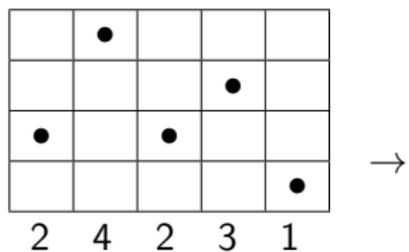
$= Q_{12}Q_{121} =$
 $Q_{12343} + Q_{13242} + Q_{14232} +$
 $Q_{23141} + Q_{24131} + Q_{34121}$



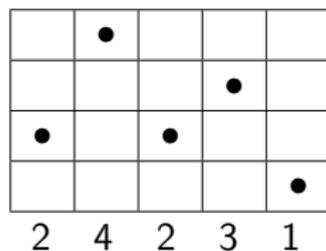
Déconcaténation



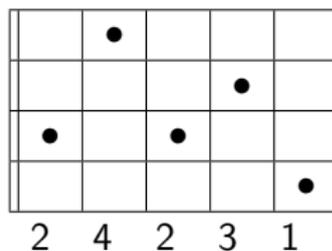
Déconcaténation



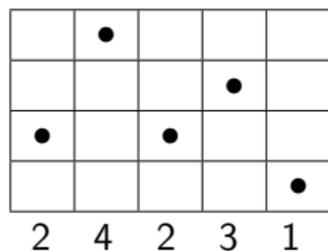
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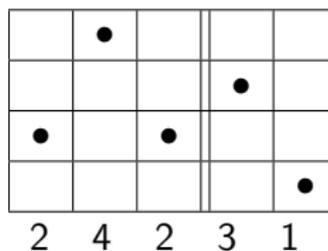
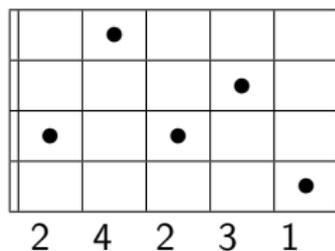
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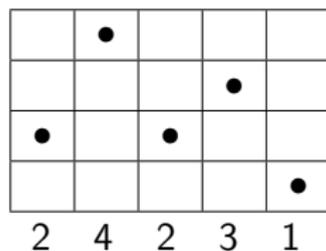
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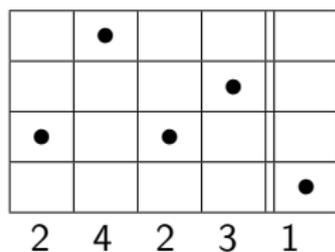
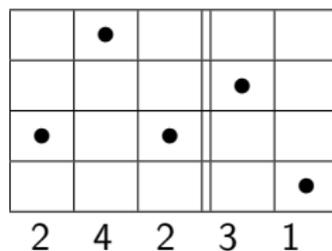
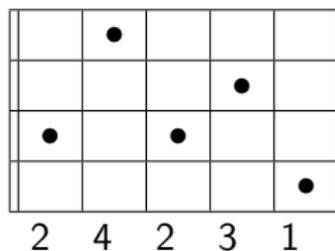
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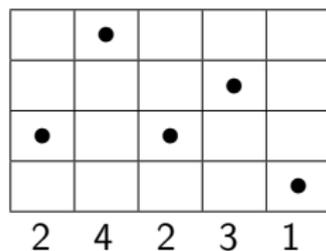
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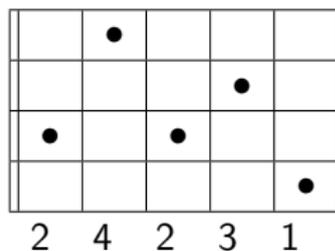
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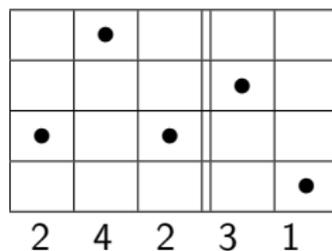
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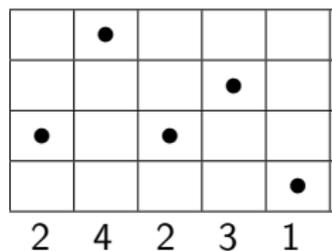
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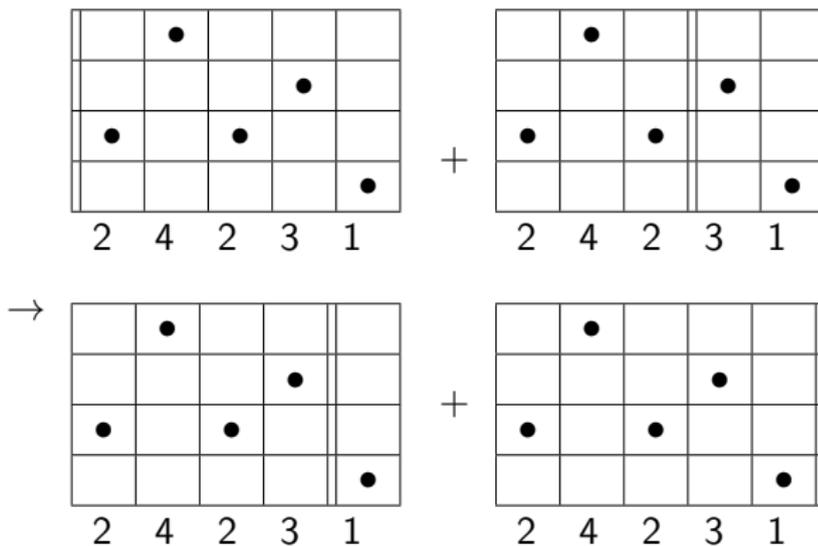
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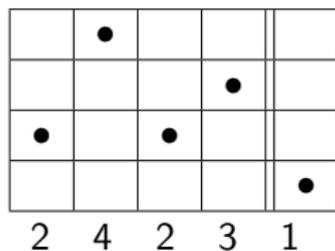
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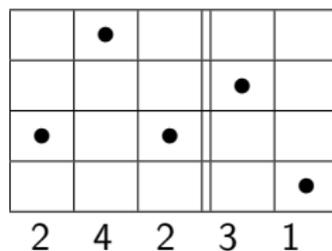
Déconcaténation

 \mathbb{R}_{24231} 

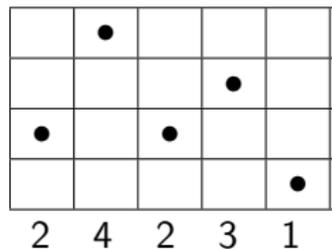
Déconcaténation

 \mathbb{R}_{24231} Δ
 \rightarrow  $\mathbb{R}_\epsilon \otimes \mathbb{R}_{24231}$

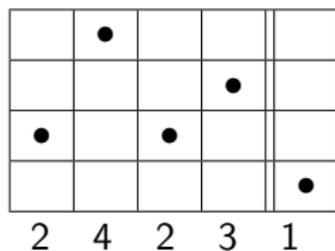
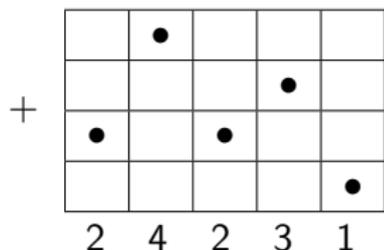
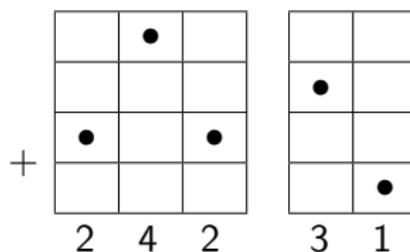
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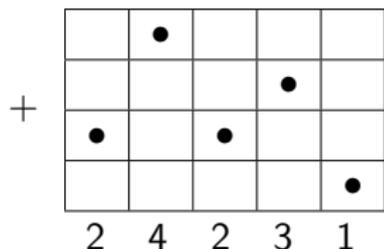
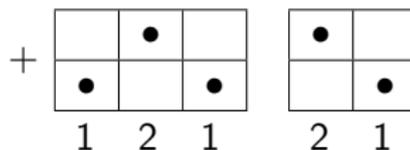
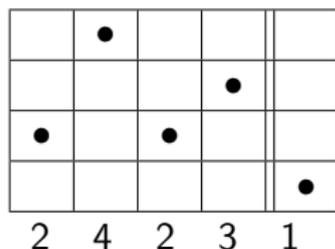
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Déconcaténation

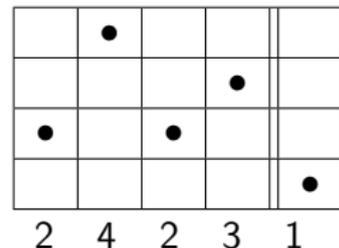
 \mathbb{R}_{24231} Δ
 \rightarrow  $\mathbb{R}_\epsilon \otimes \mathbb{R}_{24231}$ 

Déconcaténation

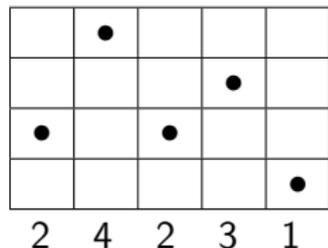
 \mathbb{R}_{24231} $\mathbb{R}_\epsilon \otimes \mathbb{R}_{24231}$ Δ
 \rightarrow 

Déconcaténation

$$\mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21}$$

 \mathbb{R}_{24231}
 Δ
 \rightarrow


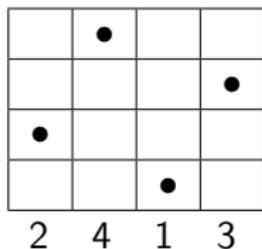
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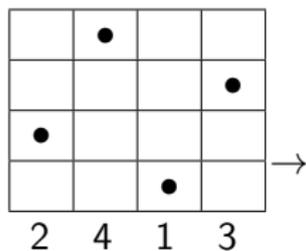
Déconcaténation

$$\begin{array}{ccc}
 & \mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} & + \mathbb{R}_{121} \otimes \mathbb{R}_{21} \\
 & \Delta & \\
 \mathbb{R}_{24231} & \rightarrow & \\
 & \mathbb{R}_{1312} \otimes \mathbb{R}_1 & + \mathbb{R}_{24231} \otimes \mathbb{R}_\epsilon
 \end{array}$$

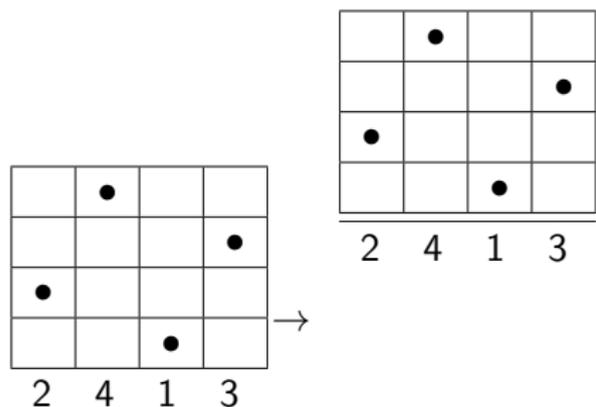
Désassemblage horizontale



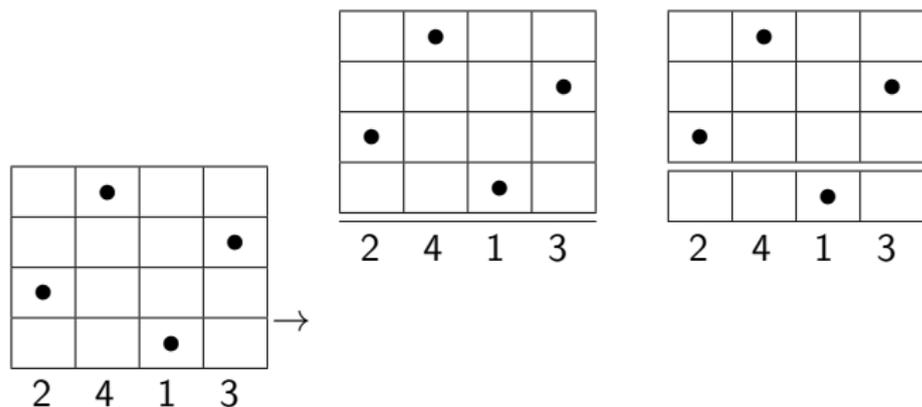
Désassemblage horizontale



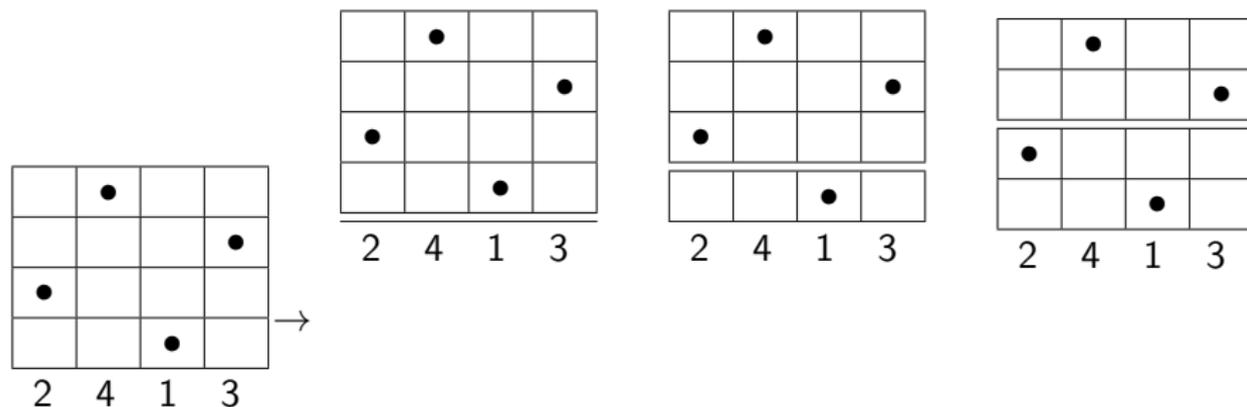
Désassemblage horizontale



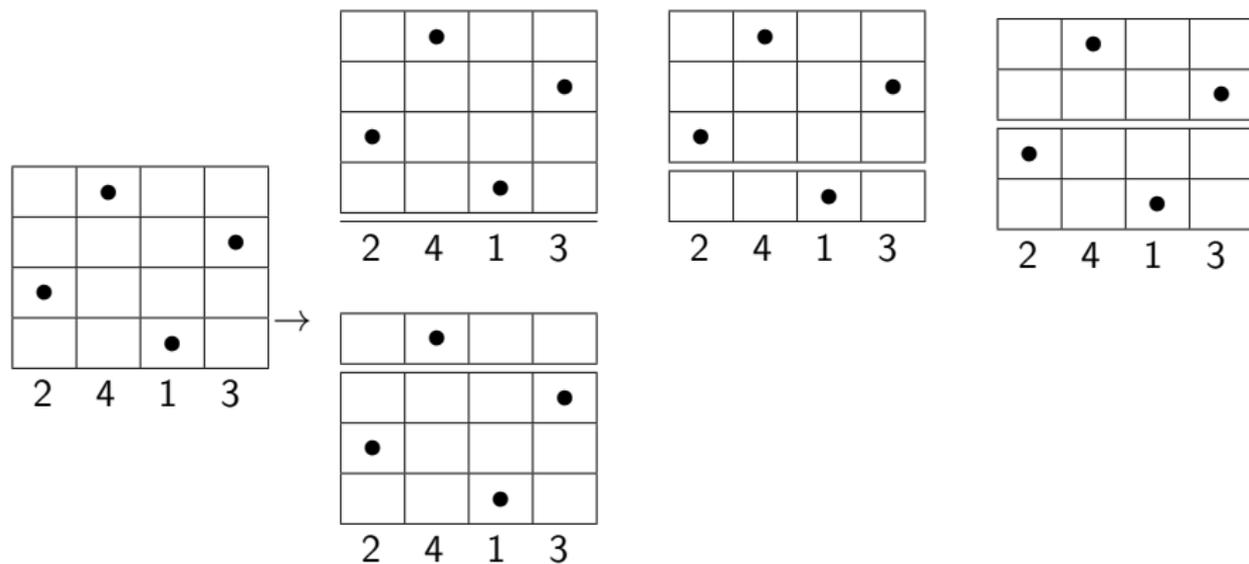
Désassemblage horizontale



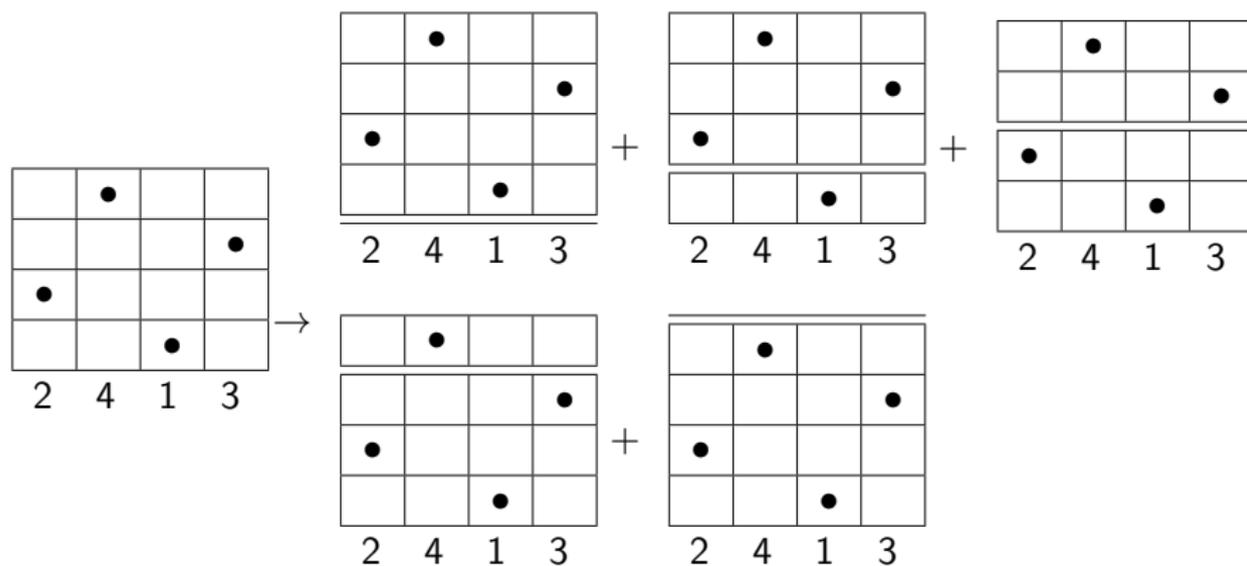
Désassemblage horizontale



Désassemblage horizontale



Désassemblage horizontale



Désassemblage horizontale

 \mathbb{Q}_{2413}

$$\begin{array}{c}
 \Delta \\
 \rightarrow
 \end{array}
 \begin{array}{c}
 \begin{array}{|c|c|c|c|}
 \hline
 & \bullet & & \\
 \hline
 & & & \bullet \\
 \hline
 \bullet & & & \\
 \hline
 & & \bullet & \\
 \hline
 \end{array}
 +
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 \end{array}
 +
 \begin{array}{|c|c|c|c|}
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 \end{array}
 \\
 \hline
 2 \quad 4 \quad 1 \quad 3
 \end{array}
 +
 \begin{array}{c}
 \begin{array}{|c|c|c|c|}
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 \end{array}
 +
 \begin{array}{|c|c|c|c|}
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 \end{array}
 \\
 \hline
 2 \quad 4 \quad 1 \quad 3
 \end{array}$$

Désassemblage horizontale

$$\mathbb{Q}_{2413} \xrightarrow{\Delta} \mathbb{Q}_{\epsilon} \otimes \mathbb{Q}_{2413}_+$$

The diagram illustrates the horizontal decomposition of the coproduct of the permutation 2413. On the left, the permutation \mathbb{Q}_{2413} is shown as a 4x4 grid with dots in the (1,2), (2,4), (3,1), and (4,3) positions. Below the grid are the labels 2, 4, 1, 3. An arrow labeled Δ points to the right, where the coproduct is shown as the sum of two permutations: $\mathbb{Q}_{\epsilon} \otimes \mathbb{Q}_{2413}_+$. The first permutation is shown as a 4x4 grid with dots in the (1,2), (2,4), (3,1), and (4,3) positions, with labels 2, 4, 1, 3 below it. The second permutation is shown as a 4x4 grid with dots in the (1,2), (2,4), (3,1), and (4,3) positions, with labels 2, 4, 1, 3 below it. The two permutations are separated by a plus sign.

Désassemblage horizontale

$$\mathbb{Q}_{2413} \xrightarrow{\Delta} \mathbb{Q}_{2413} + \mathbb{Q}_{2413} + \mathbb{Q}_{2413} + \mathbb{Q}_{2413} + \mathbb{Q}_{2413}$$

\mathbb{Q}_{2413} is represented by a 3x4 grid with dots at (1,2), (2,1), and (3,3). Below the grid are the numbers 2, 4, 1, 3.

$\mathbb{Q}_{\epsilon} \otimes \mathbb{Q}_{2413}$ is represented by a 3x4 grid with dots at (1,3), (2,4), and (3,1). Below the grid are the numbers 1, 3, 2.

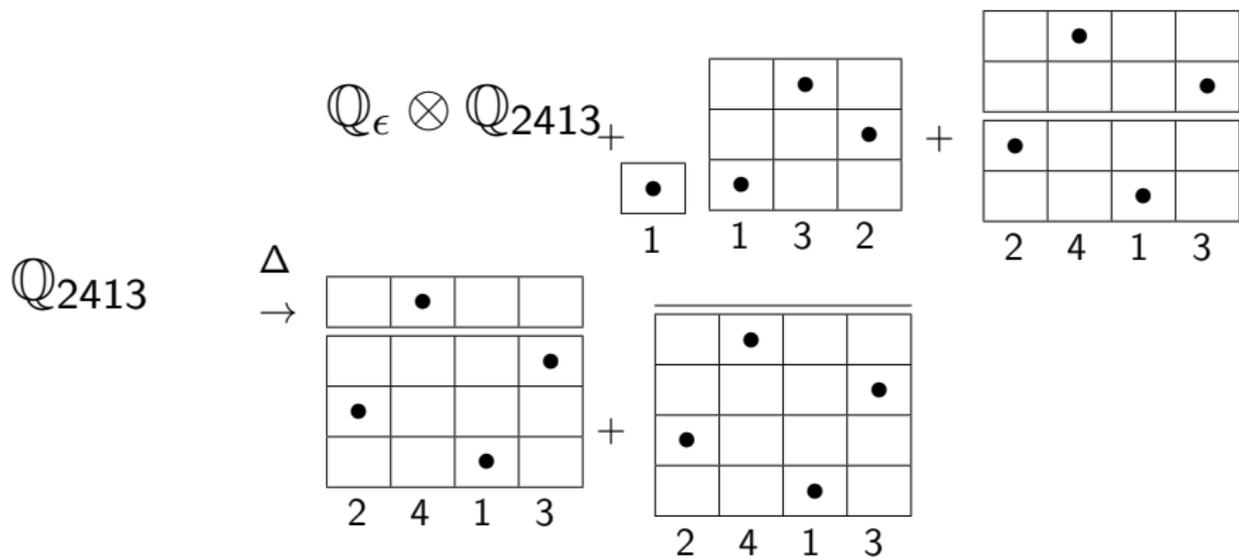
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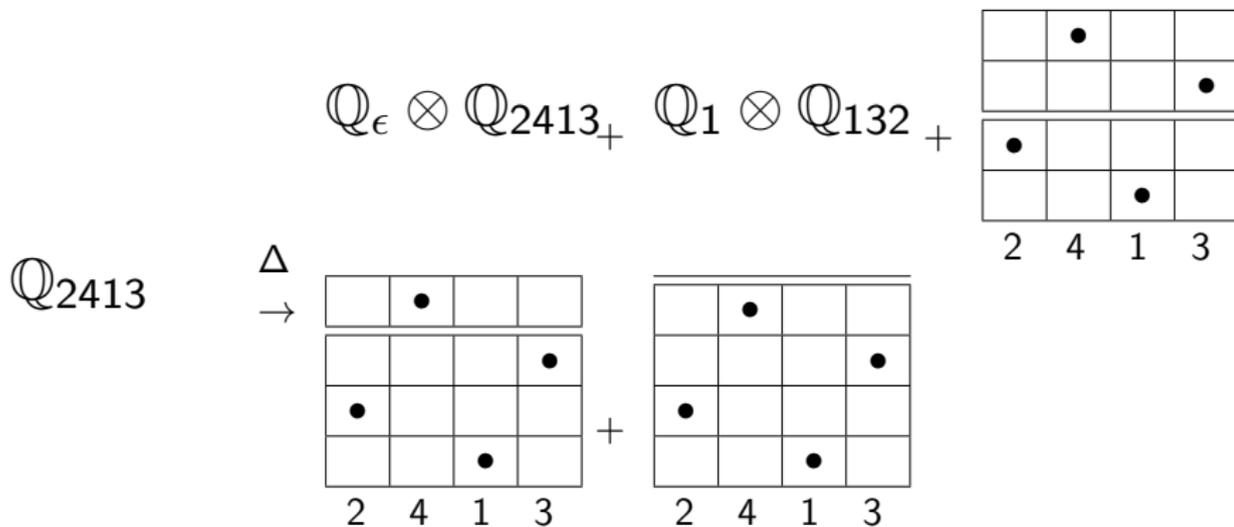
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Désassemblage horizontale



Désassemblage horizontale



Désassemblage horizontale

$$\mathbb{Q}_\epsilon \otimes \mathbb{Q}_{2413} + \mathbb{Q}_1 \otimes \mathbb{Q}_{132} + \mathbb{Q}_{21} \otimes \mathbb{Q}_{21}$$

$$\mathbb{Q}_{2413} \xrightarrow{\Delta}$$

$$\mathbb{Q}_{213} \otimes \mathbb{Q}_1 + \mathbb{Q}_{2413} \otimes \mathbb{Q}_\epsilon$$

Dualité

$$\begin{aligned} \mathbb{R}_{221}\mathbb{R}_{21} = & \mathbb{R}_{22143} + \mathbb{R}_{22413} + \mathbb{R}_{22431} + \\ & \mathbb{R}_{24213} + \mathbb{R}_{24231} + \mathbb{R}_{24321} + \\ & \mathbb{R}_{42213} + \mathbb{R}_{42231} + \mathbb{R}_{42321} + \\ & \mathbb{R}_{43221} \end{aligned}$$

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$$\begin{aligned} \Delta(\mathbb{Q}_{24231}) = & \mathbb{Q}_\epsilon \otimes \mathbb{Q}_{24231} + \mathbb{Q}_1 \otimes \mathbb{Q}_{1312} + \\ & \mathbb{Q}_{221} \otimes \mathbb{Q}_{21} + \mathbb{Q}_{2231} \otimes \mathbb{Q}_1 + \\ & \mathbb{Q}_{24231} \otimes \mathbb{Q}_\epsilon \end{aligned}$$

Dualité

$$\mathbb{Q}_{121}\mathbb{Q}_{21} = \mathbb{Q}_{12143} + \mathbb{Q}_{13142} + \mathbb{Q}_{14132} + \\ \mathbb{Q}_{23241} + \mathbb{Q}_{24231} + \mathbb{Q}_{34321}$$

$$\mathbb{R}_{221}\mathbb{R}_{21} = \mathbb{R}_{22143} + \mathbb{R}_{22413} + \mathbb{R}_{22431} + \\ \mathbb{R}_{24213} + \mathbb{R}_{24231} + \mathbb{R}_{24321} + \\ \mathbb{R}_{42213} + \mathbb{R}_{42231} + \mathbb{R}_{42321} + \\ \mathbb{R}_{43221}$$

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 &\quad \mathbb{R}_{24213} + \mathbb{R}_{24231} + \mathbb{R}_{24321} + \\
 &\quad \mathbb{R}_{42213} + \mathbb{R}_{42231} + \mathbb{R}_{42321} + \\
 &\quad \mathbb{R}_{43221} \\
 \Delta(\mathbb{Q}_{24231}) &= \mathbb{Q}_\epsilon \otimes \mathbb{Q}_{24231} + \mathbb{Q}_1 \otimes \mathbb{Q}_{1312} + \Delta(\mathbb{R}_{24231}) = \mathbb{R}_\epsilon \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \\
 &\quad \mathbb{R}_{221} \otimes \mathbb{R}_{21} + \mathbb{Q}_{2231} \otimes \mathbb{Q}_1 + \mathbb{R}_{1312} \otimes \mathbb{R}_1 + \mathbb{R}_{24231} \otimes \mathbb{R}_\epsilon \\
 &\quad \mathbb{Q}_{24231} \otimes \mathbb{Q}_\epsilon
 \end{aligned}$$

Auto-dualité

- \mathbb{R} et \mathbb{Q} bases de **WQSym** et **WQSym***

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Demis produits

Définition récursive du produit de mélange

- $\epsilon \sqcup w = w \sqcup \epsilon = w$
- $ua \sqcup vb = (u \sqcup vb)a + (ua \sqcup v)b$

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Exemple de produits gauche et droit

- $\mathbb{R}_{12}\mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$

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Exemple de coproduits gauche et droit

- $\tilde{\Delta}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321} + \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_1$

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Exemple de coproduits gauche et droit

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- $\Delta_{\prec}(\mathbb{R}_{2425531}) = \mathbb{R}_{12133} \otimes \mathbb{R}_{21} + \mathbb{R}_{131442} \otimes \mathbb{R}_1$
- $\Delta_{\succ}(\mathbb{R}_{2425531}) = \mathbb{R}_{121} \otimes \mathbb{R}_{3321}$

Bigèbre bidendriforme

Définition

- Raffinement de l'associativité et la coassociativité
 $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$

Bigèbre bidendriforme

Définition

- Raffinement de l'associativité et la coassociativité
 - $(a \prec b) \prec c = a \prec (b \prec c + b \succ c),$
 - $(a \succ b) \prec c = a \succ (b \prec c),$
 - $(a \prec b + a \succ b) \succ c = a \succ (b \succ c).$

Bigèbre bidendriforme

Définition

- Raffinement de l'associativité et la coassociativité
 - 3 et 3 équations

Bigèbre bidendriforme

Définition

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 - 4 équations

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Théorème [Foissy]

Si A est une bigèbre bidendriforme alors A est généré librement par $\text{TPrim}(A)$ en tant qu'algèbre dendriforme.

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Séries

n	1	2	3	4	5	6	7	8
WQSym_n	1	3	13	75	541	4 683	47 293	545 835
TPrim _n	1	1	4	28	240	2 384	26 832	337 168

Bigèbre bidendriforme

Définition

- Raffinement de l'associativité et la coassociativité
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Si A est une bigèbre bidendriforme alors A est généré librement par $\text{TPrim}(A)$ en tant qu'algèbre dendriforme.

Corollaire

WQSym est auto-duale.

Définitions

Élément primitif

P est un éléments primitif $\iff \tilde{\Delta}(P) = 0$

Ex : $\mathbb{R}_{1213} - \mathbb{R}_{2321}$

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Élément totalement primitif

P est une élément totalement primitif $\iff \Delta_{<}(P) = \Delta_{>}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

Mon but

Isomorphisme bidendriforme explicite entre **WQSym** et sa duale

Mon but

Isomorphisme bidendriforme explicite entre **WQSym** et sa duale
 \Updownarrow
Isomorphisme explicite entre $\text{TPrim}(\mathbf{WQSym})$ et le dual

Mon but

Isomorphisme bidendriforme explicite entre **WQSym** et sa duale

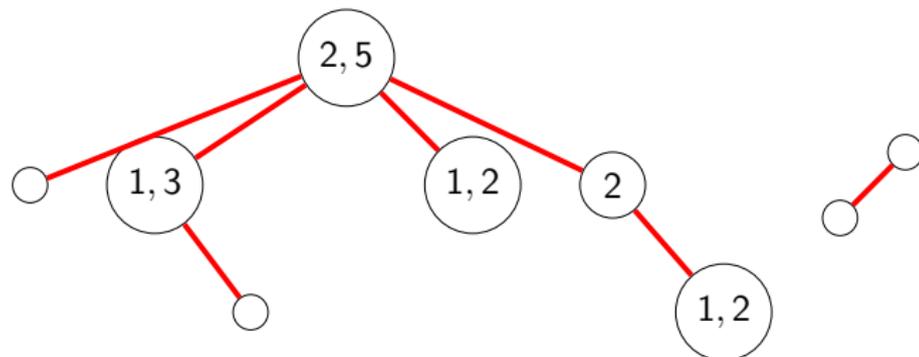


Isomorphisme explicite entre $\text{TPrim}(\mathbf{WQSym})$ et le dual

Construction de deux bases de totalement primitif
(dans **WQSym** et **WQSym**^{*})

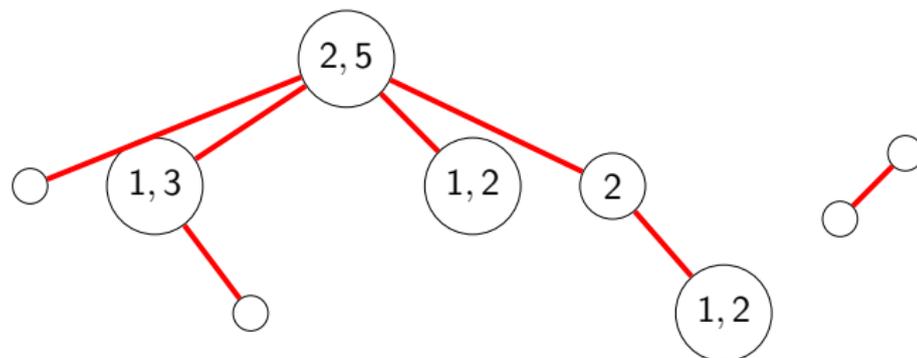
Forêts biplanes décorés

Un exemple de forêt biplane décorée:



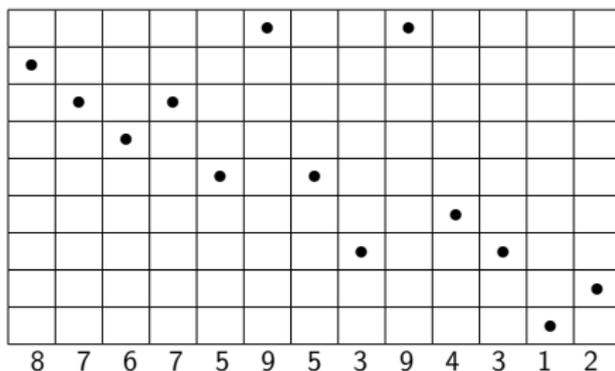
Forêts biplanes décorés

Un exemple de forêt biplane décorée:



En bijection avec: 8767595394312.

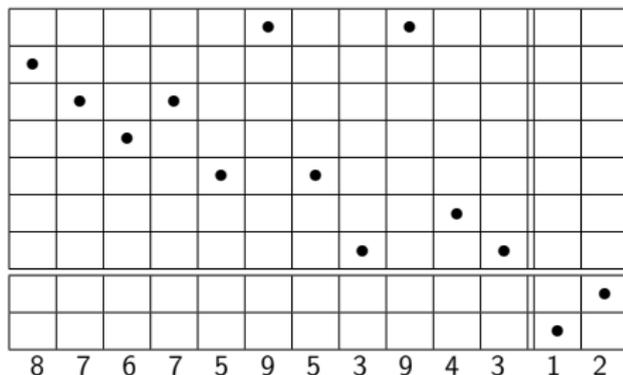
Algorithme sur 8767595394312

$$F(8767595394312)$$


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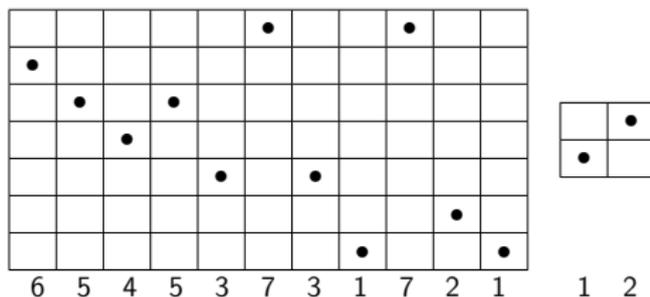
Factorisation en descentes
globales



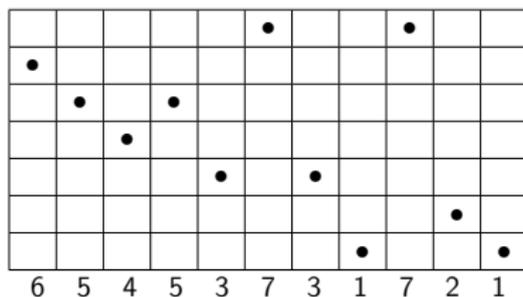
Algorithme sur 8767595394312

$$F(8767595394312) = \\ T(65453731721)T(12)$$

Factorisation en descentes
globales + tassement



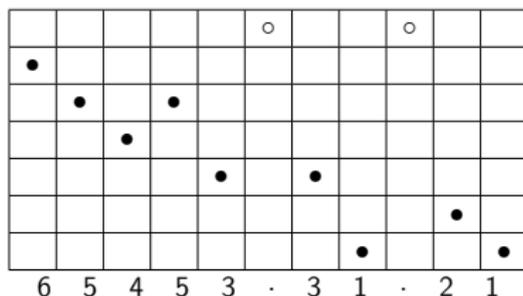
Algorithme sur 8767595394312

$$T(65453731721)T(12)$$


Algorithme sur 8767595394312

Retrait des lettres de valeur
max

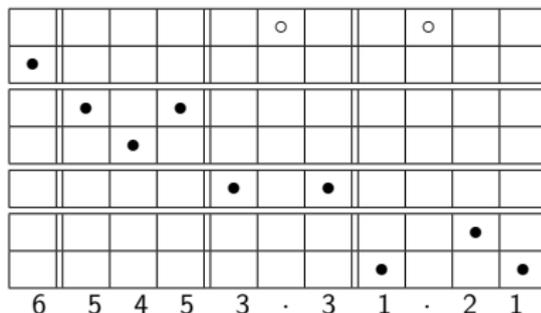
$T(65453731721)T(12)$



Algorithmme sur 8767595394312

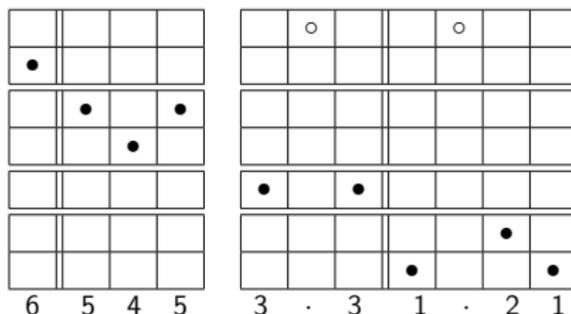
Factorisation en descentes
globales

$$T(65453731721)T(12)$$



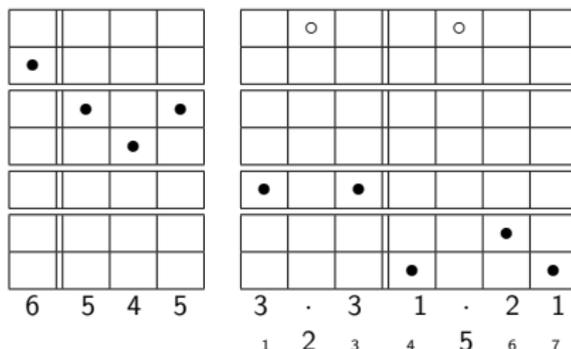
Algorithme sur 8767595394312

Distinction de deux groupes de facteurs

$$T(65453731721)T(12)$$


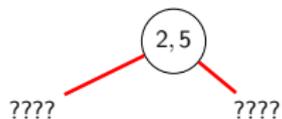
Algorithme sur 8767595394312

Positions des max

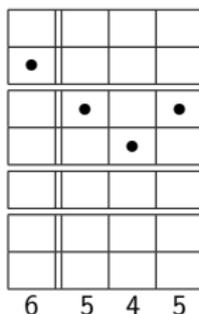
 $T(65453731721)T(12)$


Algorithme sur 8767595394312

$$T(65453731721)T(12) =$$



$$T(12)$$

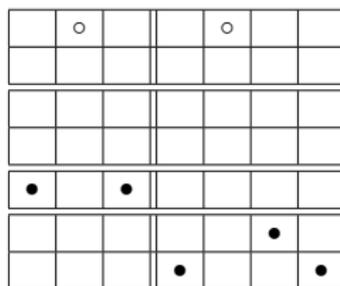


6

5

4

5



3

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3

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2

1

1

2

3

4

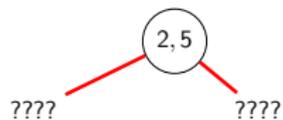
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6

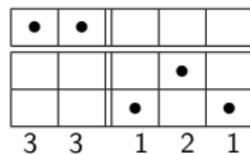
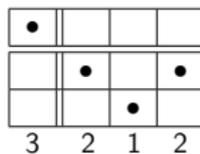
7

Algorithme sur 8767595394312

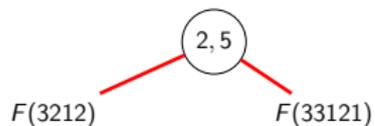
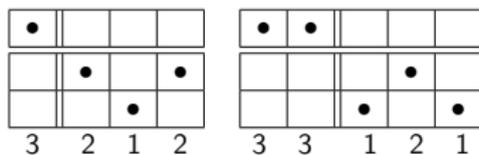
$$T(65453731721)T(12) =$$



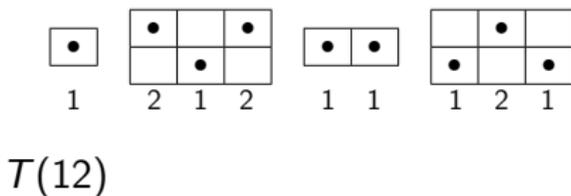
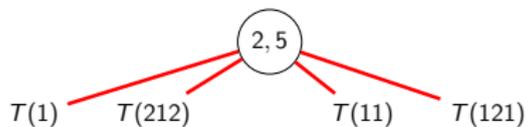
$$T(12)$$



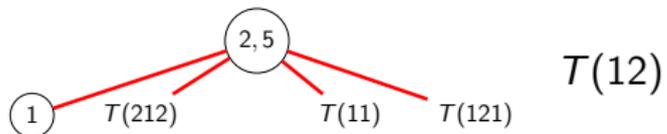
Algorithme sur 8767595394312


 $T(12)$


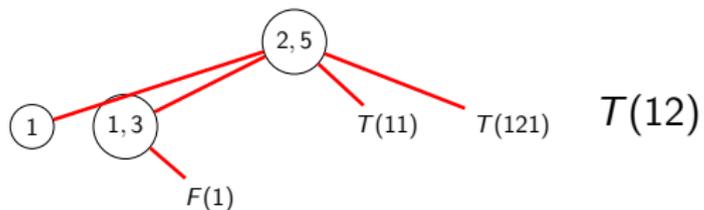
Algorithme sur 8767595394312



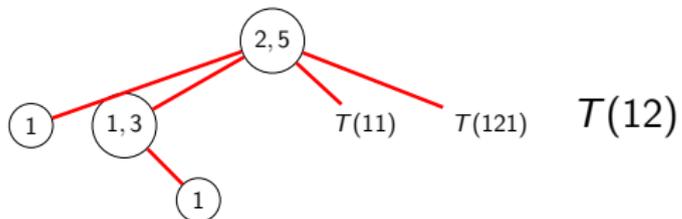
Algorithme sur 8767595394312



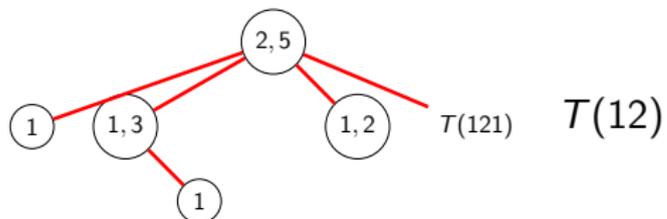
Algorithme sur 8767595394312



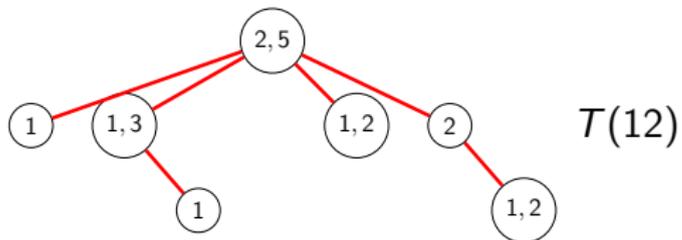
Algorithme sur 8767595394312



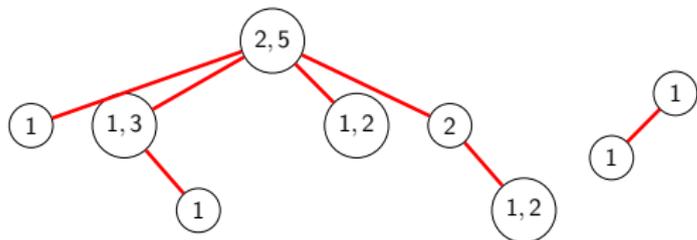
Algorithme sur 8767595394312



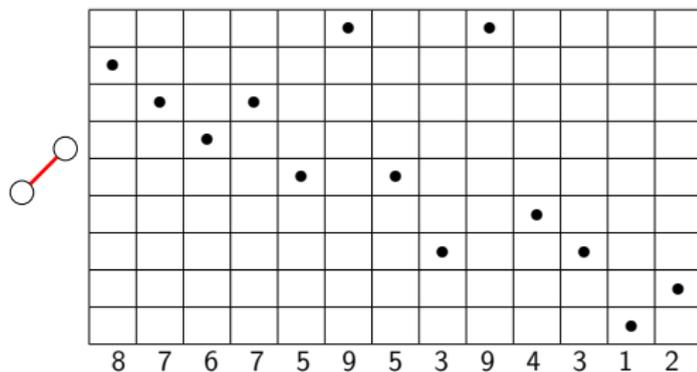
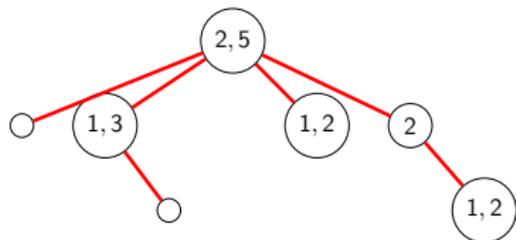
Algorithme sur 8767595394312



Algorithme sur 8767595394312



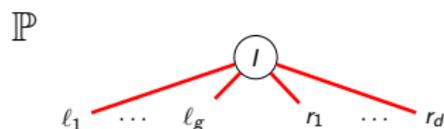
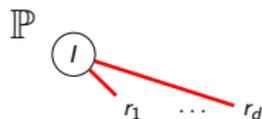
Algorithme sur 8767595394312



La base \mathbb{P}

$$\mathbb{P}_{\circ} := \mathbb{R}_1,$$

$$\begin{aligned} \mathbb{P}_{t_1, \dots, t_k} &:= (\dots (\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1}, \\ &:= \Phi_I(\mathbb{P}_{r_1, \dots, r_d}), \end{aligned}$$

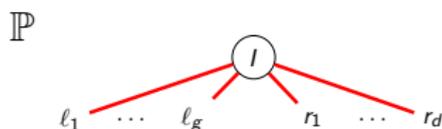
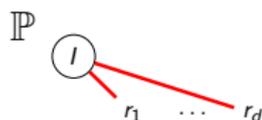


$$:= \langle \mathbb{P}_{l_1}, \mathbb{P}_{l_2}, \dots, \mathbb{P}_{l_k}; \Phi_I(\mathbb{P}_{r_1, \dots, r_d}) \rangle.$$

La base \mathbb{P}

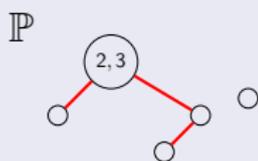
$$\mathbb{P}_{\circ} := \mathbb{R}_1,$$

$$\begin{aligned} \mathbb{P}_{t_1, \dots, t_k} &:= (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1}, \\ &:= \Phi_I(\mathbb{P}_{r_1, \dots, r_d}), \end{aligned}$$



$$:= \langle \mathbb{P}_{l_1}, \mathbb{P}_{l_2}, \dots, \mathbb{P}_{l_k}; \Phi_I(\mathbb{P}_{r_1, \dots, r_d}) \rangle.$$

Exemple

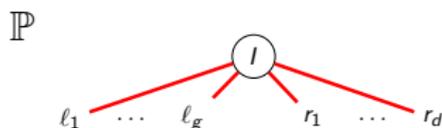
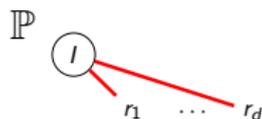


$$\begin{aligned} &= \mathbb{R}_{23541} - \mathbb{R}_{245531} - \mathbb{R}_{244531} - \mathbb{R}_{245431} - \\ &\quad \mathbb{R}_{254431} + \mathbb{R}_{325541} - \mathbb{R}_{425531} - \mathbb{R}_{524431} + \\ &\quad \mathbb{R}_{352541} - \mathbb{R}_{452531} + \mathbb{R}_{355241} - \mathbb{R}_{455231} + \\ &\quad \mathbb{R}_{344521} + \mathbb{R}_{345421} + \mathbb{R}_{354421} + \mathbb{R}_{534421} \end{aligned}$$

La base \mathbb{P}

$$\mathbb{P}_{\circ} := \mathbb{R}_1,$$

$$\begin{aligned} \mathbb{P}_{t_1, \dots, t_k} &:= (\dots (\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1}, \\ &:= \Phi_I(\mathbb{P}_{r_1, \dots, r_d}), \end{aligned}$$

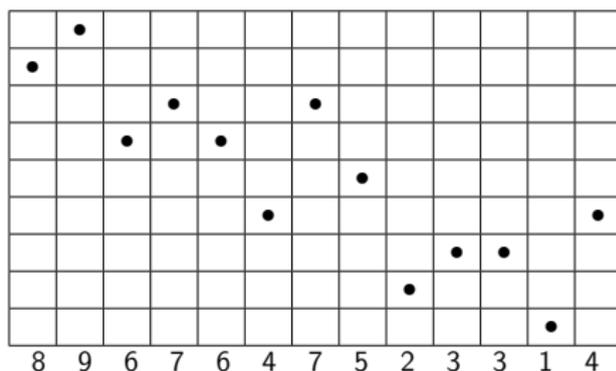


$$:= \langle \mathbb{P}_{l_1}, \mathbb{P}_{l_2}, \dots, \mathbb{P}_{l_k}; \Phi_I(\mathbb{P}_{r_1, \dots, r_d}) \rangle.$$

Théorème

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_n}$ est une base de \mathbf{WQSym}_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_n}$ est une base de \mathbf{Prim}_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{TP}_n}$ est une base de \mathbf{TPrim}_n .

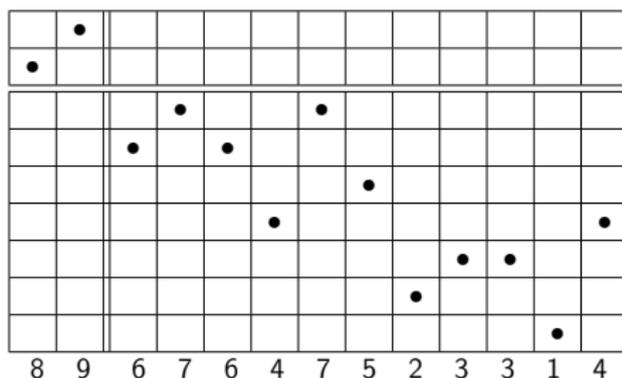
Algorithme sur 8967647523314

$$F^*(8967647523314)$$


Algorithme sur 8967647523314

Factorisation en descentes globales

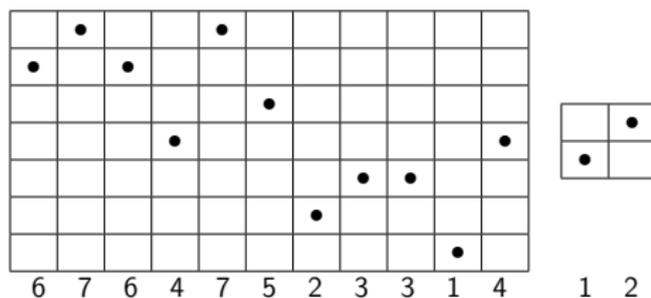
$F^*(8967647523314)$



Algorithme sur 8967647523314

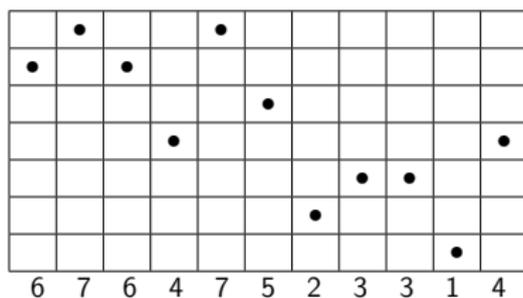
$$F^*(8967647523314) = \\ T^*(67647523314)T^*(12)$$

Factorisation en descentes globales
+ tassement + échange



Algorithme sur 8967647523314

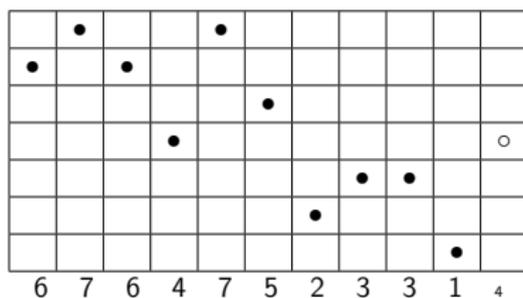
$$T^*(67647523314)T^*(12)$$



Algorithme sur 8967647523314

Retrait de la dernière lettre

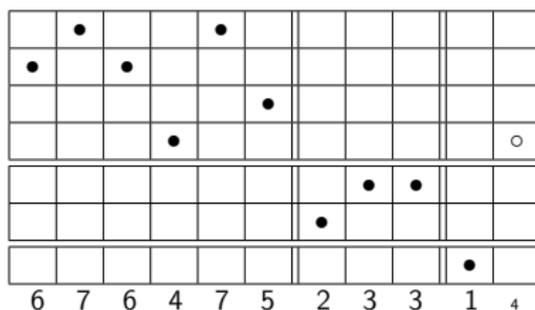
$$T^*(67647523314)T^*(12)$$



Algorithme sur 8967647523314

Factorisation en descentes globales

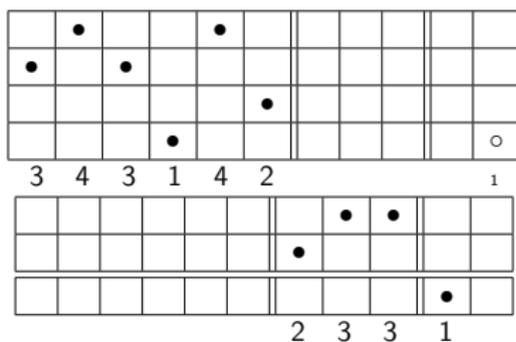
$$T^*(67647523314)T^*(12)$$



Algorithme sur 8967647523314

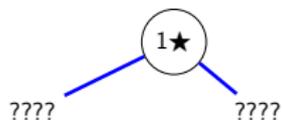
$$T^*(67647523314)T^*(12)$$

Distinction de deux groupes de facteurs

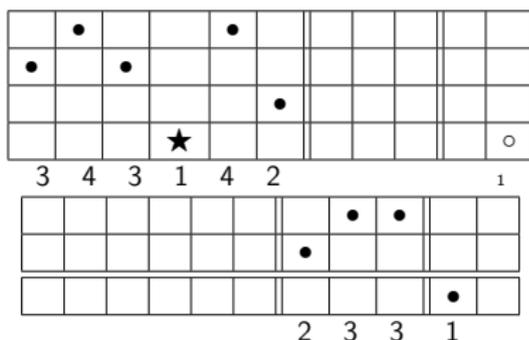


Algorithme sur 8967647523314

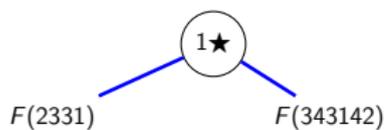
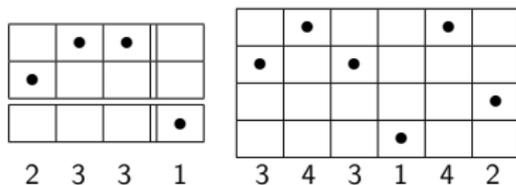
$$T^*(67647523314) T^*(12) =$$


 $T^*(12)$

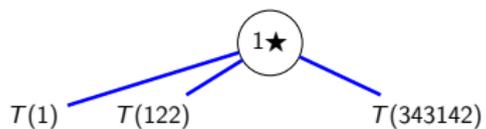
La dernière lettre est-elle présente dans le reste du mot ?



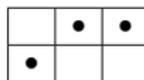
Algorithme sur 8967647523314


 $T^*(12)$


Algorithme sur 8967647523314


 $T^*(12)$

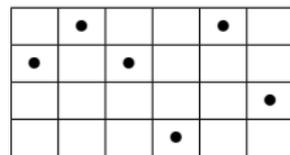

1



1

2

2



3

4

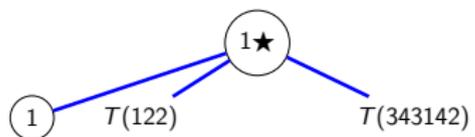
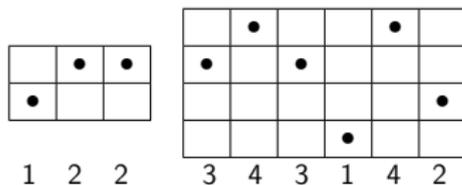
3

1

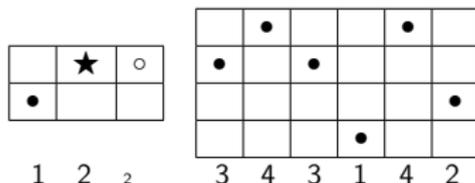
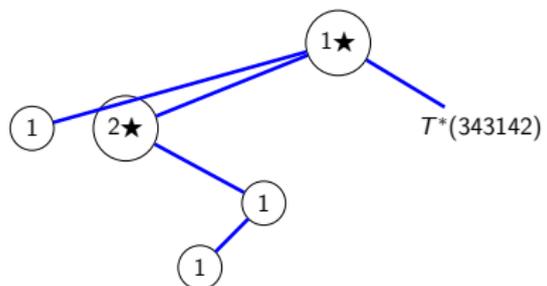
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2

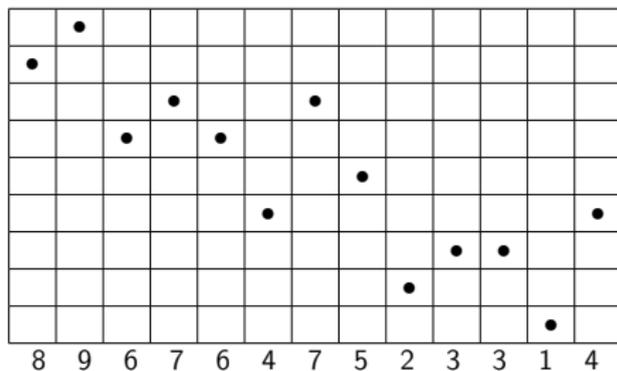
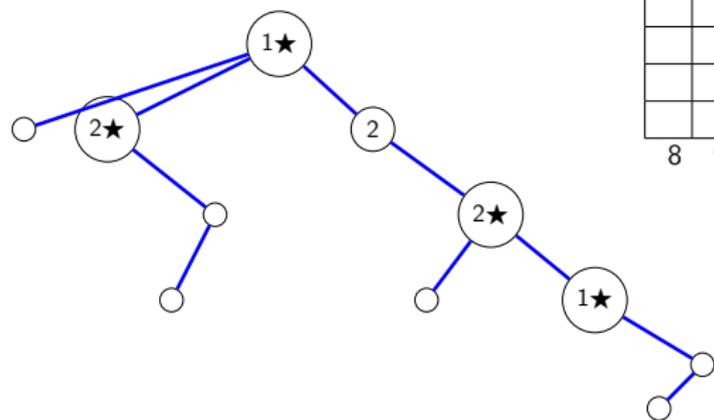
Algorithme sur 8967647523314


 $T^*(12)$


Algorithme sur 8967647523314

 $T^*(12)$

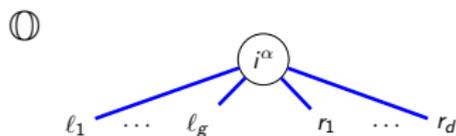
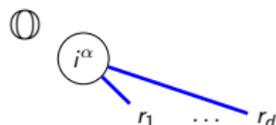
Algorithme sur 8967647523314



La base \mathbb{O}

$$\mathbb{O}_{\circ} := \mathbb{Q}_1,$$

$$\begin{aligned} \mathbb{O}_{t_1, \dots, t_k} &:= (\dots (\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1}, \\ &:= \Psi_i^\alpha(\mathbb{O}_{r_1, \dots, r_d}), \end{aligned}$$

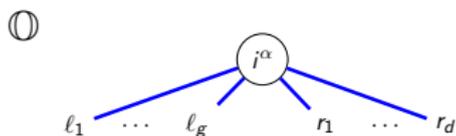
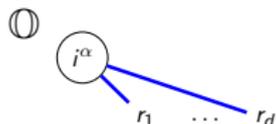


$$:= \langle \mathbb{O}_{l_1}, \mathbb{O}_{l_2}, \dots, \mathbb{O}_{l_k}; \Psi_i^\alpha(\mathbb{O}_{r_1, \dots, r_d}) \rangle.$$

La base \mathbb{O}

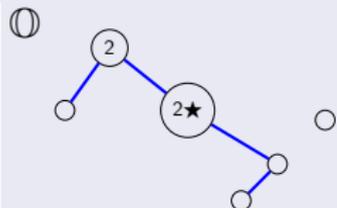
$$\mathbb{O}_{\circ} := \mathbb{Q}_1,$$

$$\begin{aligned} \mathbb{O}_{t_1, \dots, t_k} &:= (\dots (\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1}, \\ &:= \Psi_i^\alpha(\mathbb{O}_{r_1, \dots, r_d}), \end{aligned}$$



$$:= \langle \mathbb{O}_{l_1}, \mathbb{O}_{l_2}, \dots, \mathbb{O}_{l_g}; \Psi_i^\alpha(\mathbb{O}_{r_1, \dots, r_d}) \rangle.$$

Exemple



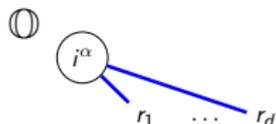
=

$$\begin{aligned} &\mathbb{Q}_{531442} + \mathbb{Q}_{521443} + \mathbb{Q}_{512443} - \mathbb{Q}_{534142} - \\ &\mathbb{Q}_{524143} - \mathbb{Q}_{514243} - \mathbb{Q}_{514432} - \mathbb{Q}_{524431} - \\ &\mathbb{Q}_{514423} + \mathbb{Q}_{541432} + \mathbb{Q}_{542431} + \mathbb{Q}_{541423} \end{aligned}$$

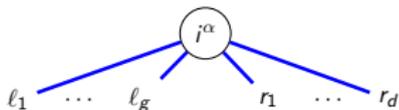
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 \mathbb{O}

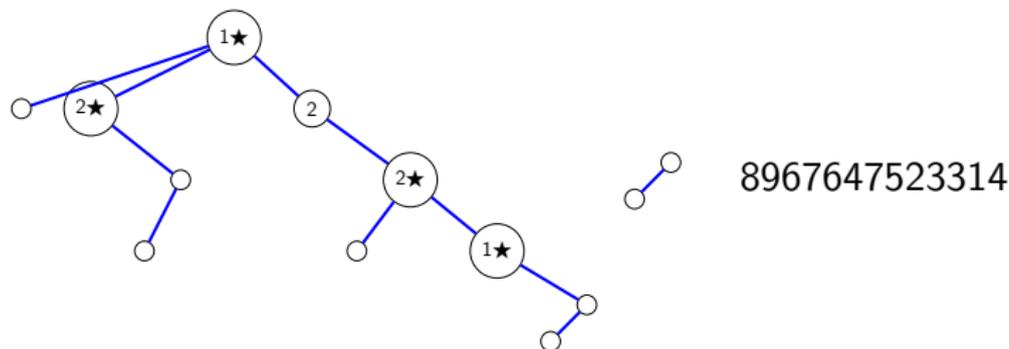
$$:= \langle \mathbb{O}_{l_1}, \mathbb{O}_{l_2}, \dots, \mathbb{O}_{l_k}; \Psi_i^\alpha(\mathbb{O}_{r_1, \dots, r_d}) \rangle.$$



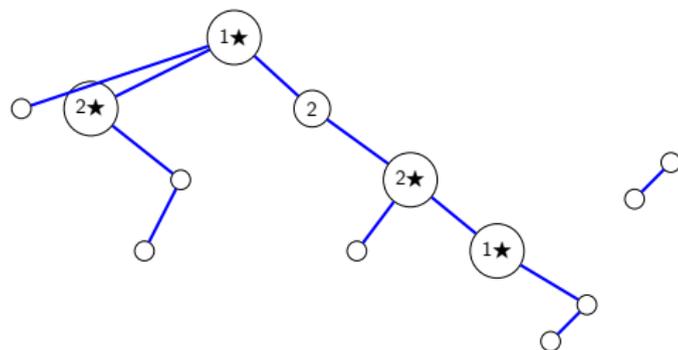
Théorème

- $(\mathbb{O}_f)_{f \in \mathfrak{F}_n^*}$ est une base de \mathbf{WQSym}_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{T}_n^*}$ est une base de \mathbf{Prim}_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{P}_n^*}$ est une base de \mathbf{TPrim}_n^* .

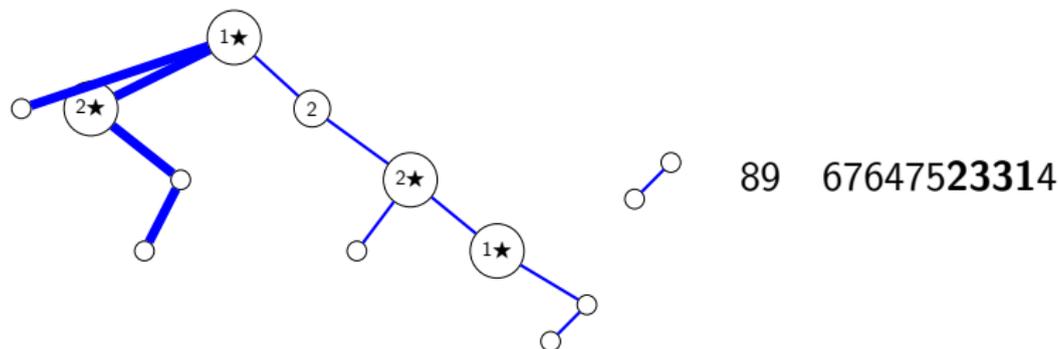
Une bijection

 F $[1 - 9]^{13}$

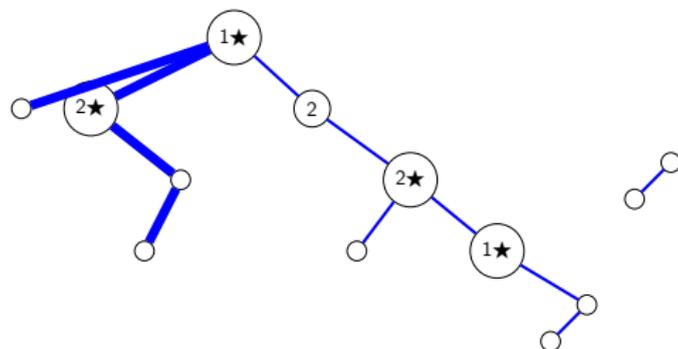
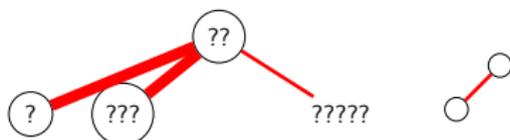
Une bijection

**89** 67647523314 T_1 T_2 $[3 - 9]^{11}$ $[1, 2]^2$

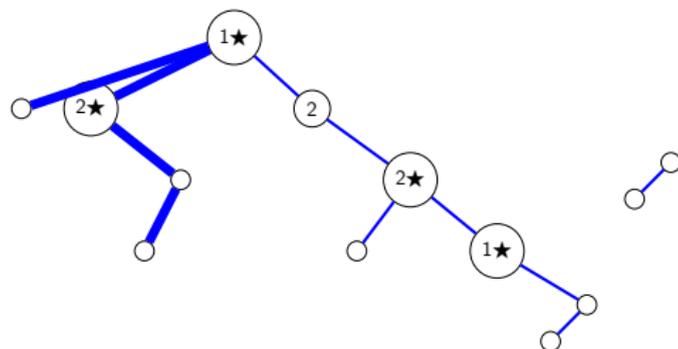
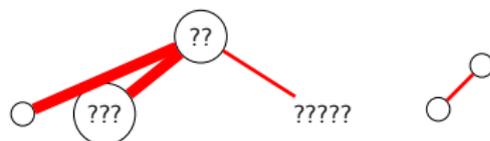
Une bijection

 T_1 T_2
 $[3 - 9]^{11} \quad [1, 2]^2$

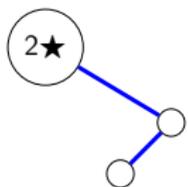
Une bijection

89 676475**23314** $[6 - 8]^4 [3 - 5, 9]^7 12$

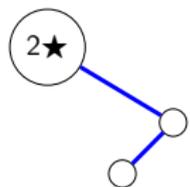
Une bijection

89 676475**233**148[**6, 7**]³[3 – 5, 9]⁷12

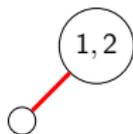
Une bijection

 122  $[1, 2]^3$

Une bijection

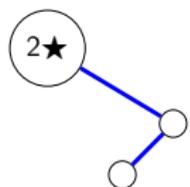


122

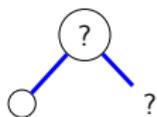
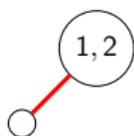
 $[1, 2]^3$  $[1, 2]^3$ 

122

Une bijection

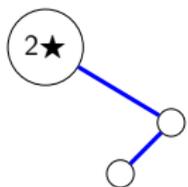


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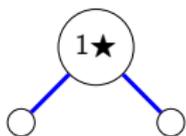
 $[1, 2]^3$  $[1, 2]^3$ 

122

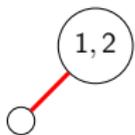
Une bijection



122

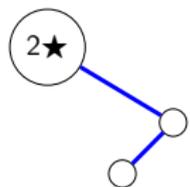
 $[1, 2]^3$ 

212

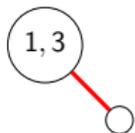


122

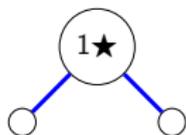
Une bijection



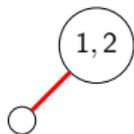
122



212

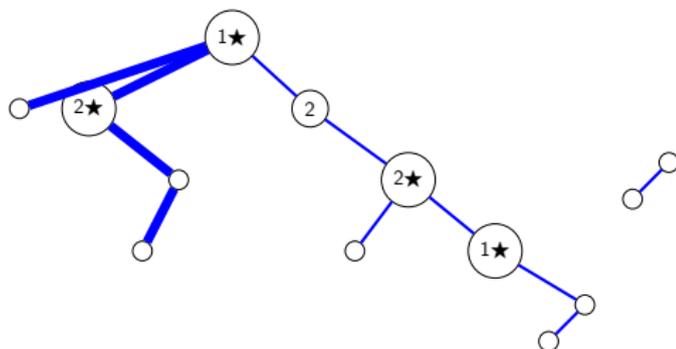
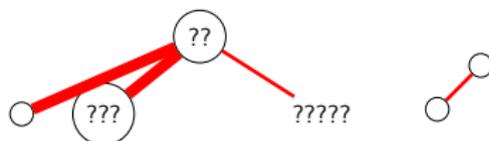


212

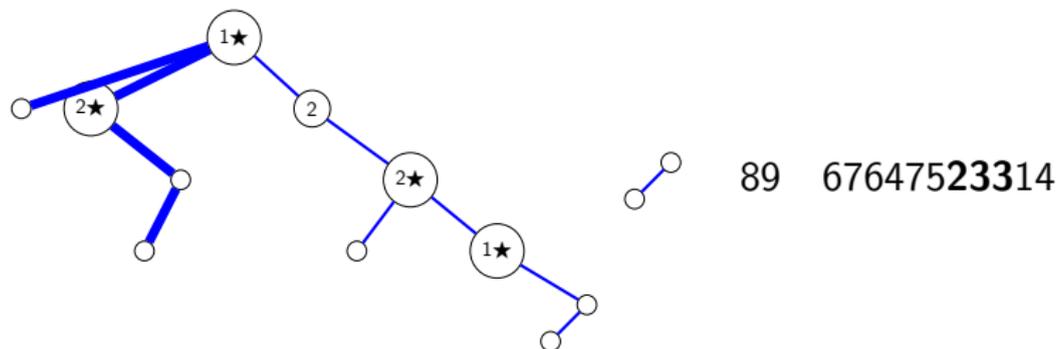


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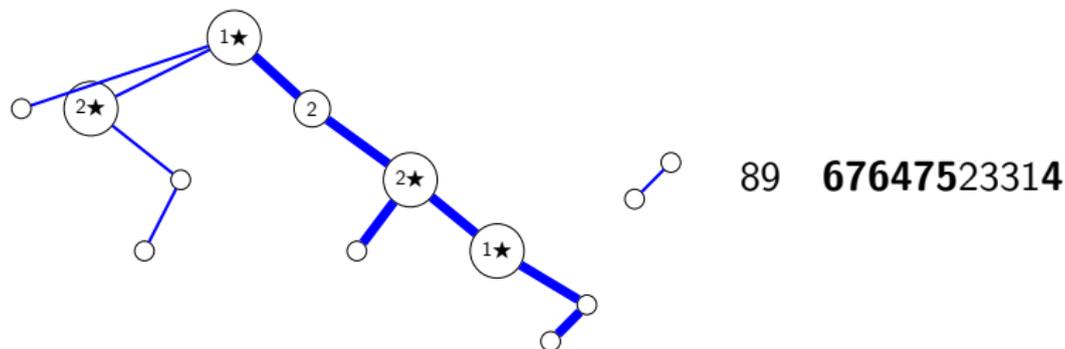
Une bijection

89 676475**233**14 $8[6, 7]^3[3 - 5, 9]^7 12$

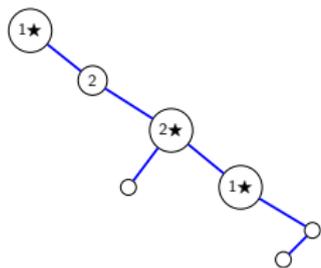
Une bijection



Une bijection



Une bijection

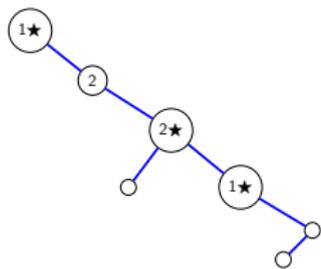


$$3431421 = \text{pack}(6764754)$$

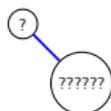
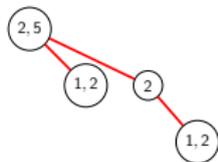


$$[1 - 4]^7 = \text{pack}([3 - 5, 9]^7)$$

Une bijection

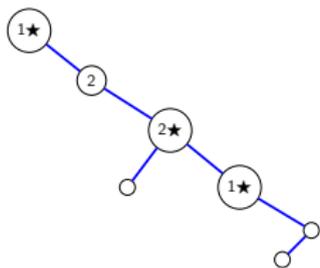


3431421

 $[1 - 4]^7$  $[1 - 4]^7$ 

3431421

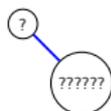
Une bijection



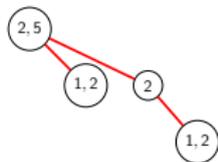
$$3431421 = \text{pack}(6764754)$$



$$[1 - 4]^7 = \text{pack}([3 - 5, 9]^7)$$

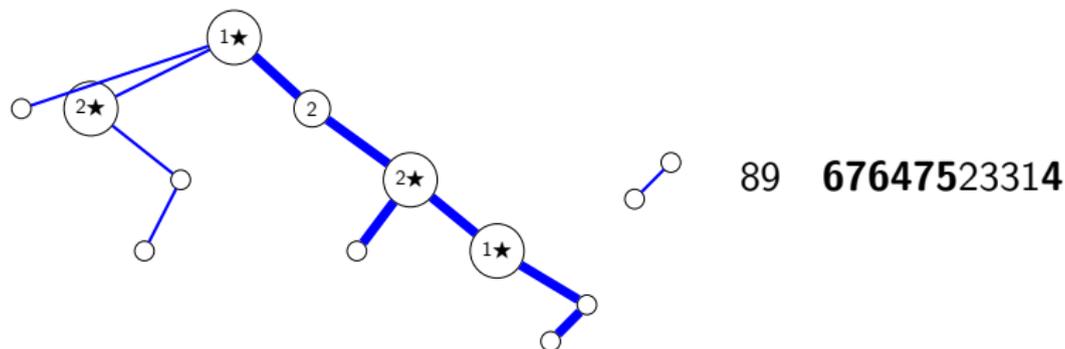


$$[1 - 4]^7 = \text{pack}([4 - 7]^7)$$

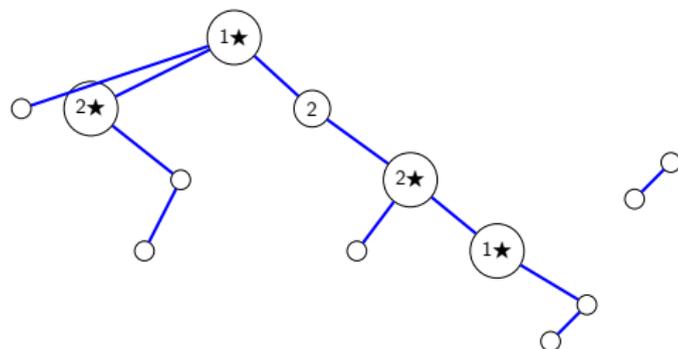


$$3431421 = \text{pack}(5953943)$$

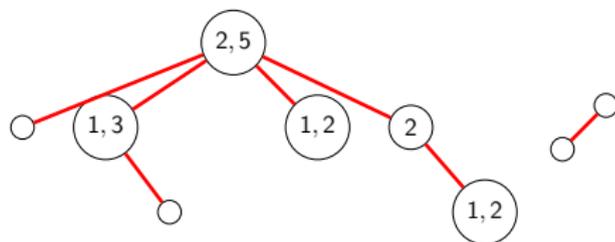
Une bijection



Une bijection



8967647523314



8767595394312

Théorèmes

Théorème

- $(\mathbb{O}_f)_{f \in \mathfrak{F}_n^*}$ base de \mathbf{WQSym}_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{T}_n^*}$ base de Prim_n^* ,
- $(\mathbb{O}_t)_{t \in \mathfrak{P}_n^*}$ base de TPrim_n^* .

Théorème

- $(\mathbb{P}_f)_{f \in \mathfrak{F}_n}$ base de \mathbf{WQSym}_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{T}_n}$ base de Prim_n ,
- $(\mathbb{P}_t)_{t \in \mathfrak{P}_n}$ base de TPrim_n .

Bijection

$$\mathfrak{F}^* \leftrightarrow \mathfrak{F}$$

$$\mathfrak{T}^* \leftrightarrow \mathfrak{T}$$

$$\mathfrak{P}^* \leftrightarrow \mathfrak{P}$$

Isomorphisme bidendriforme