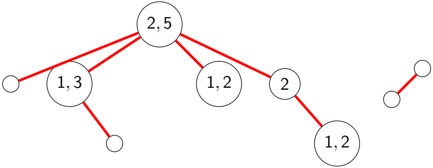
Basis of totally primitive elements of **WQSym**

Thanks for coming! These slides are made in such a way that you can jump around using the links in green (ex: Go to main result) or in the bottom-right of every slide. You will also find some clickable ❓ where I can give you more details, don't hesitate to ask !



$\longleftarrow$ ❓ $\longrightarrow$ 8767595394312

Problematic, Main result and Steps

- 1999 Hivert define **WQSym** the Hopf algebra on
  packed words/surjections/ordered set partitions,
- 2001 Duchanp-Hivert-Thibon conjecture auto-duality of **WQSym**,
- 2005 Foissy proves this auto-duality with bidendriform structure.

Main Result

An explicit bidendriform morphism from **WQSym** to $\mathbf{WQSym}^*$.

Steps to construct the automorphism:

- Basis $\mathbb{P}$ of totally primitive elements of **WQSym**,
- Basis $\mathbb{O}$ of totally primitive elements of $\mathbf{WQSym}^*$,
- Bijection between biplan forests.

Packed words

Definition

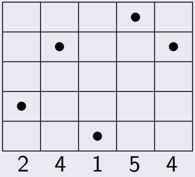
A word over the alphabet $\mathbb{N}_{>0}$ is packed if all the letters from 1 to its maximum *m* appears at least once.

Packed words of size 0, 1, 2 and 3

- ϵ
- 1
- 12 21 11
- 123 132 213 231 312 321
  122 212 221 112 121 211 111

The function *pack*

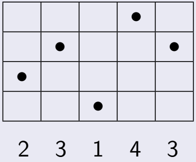
$24154 \notin$ **PW**



remove empty lines

$\rightarrow$ pack $\rightarrow$

pack$(24154) = 23143 \in$ **PW**



WQSym, the Hopf algebra on packed words

WQSym

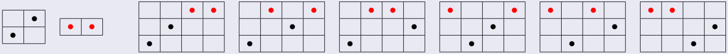
- $\mathbb{R}_{3112} + \mathbb{R}_{212} - 3\mathbb{R}_{212341} - \frac{5}{3}\mathbb{R}_{111}$
- $\mathbb{R}_{12} \cdot \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\Delta(\mathbb{R}_{24231}) = \mathbb{R}_{\epsilon} \otimes \mathbb{R}_{24231} + \mathbb{R}_{121} \otimes \mathbb{R}_{21} + \mathbb{R}_{1312} \otimes \mathbb{R}_{1} + \mathbb{R}_{24231} \otimes \mathbb{R}_{\epsilon}$

- Formal sums of packed words
- An associative and unitary product $\cdot$
- A coassociative and counitary coproduct Δ
- The Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$

Basis $\mathbb{R}$ of **WQSym**

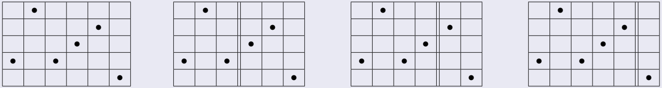
Product on $\mathbb{R}$: Shifted shuffle

$$\mathbb{R}_{12} \cdot \mathbb{R}_{\color{red}{11}} = \mathbb{R}_{12\color{red}{33}} + \mathbb{R}_{1\color{red}{3}2\color{red}{3}} + \mathbb{R}_{1\color{red}{33}2} + \mathbb{R}_{\color{red}{3}12\color{red}{3}} + \mathbb{R}_{\color{red}{3}1\color{red}{3}2} + \mathbb{R}_{\color{red}{33}12}$$



Reduced coproduct on $\mathbb{R}$: Deconcatenation forbidding cut between two equal letters

$$\tilde{\Delta}(\mathbb{R}_{252341}) = \mathbb{R}_{121} \otimes \mathbb{R}_{231} + \mathbb{R}_{1312} \otimes \mathbb{R}_{21} + \mathbb{R}_{14123} \otimes \mathbb{R}_{1}$$



Bidendriform bialgebra

Example of half products

- $\mathbb{R}_{12} \cdot \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{1332} + \mathbb{R}_{3123} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\mathbb{R}_{12} \prec \mathbb{R}_{11} = \mathbb{R}_{1332} + \mathbb{R}_{3132} + \mathbb{R}_{3312}$
- $\mathbb{R}_{12} \succ \mathbb{R}_{11} = \mathbb{R}_{1233} + \mathbb{R}_{1323} + \mathbb{R}_{3123}$

Example of half coproducts

- $\tilde{\Delta}(\mathbb{R}_{242\mathbf{55}31}) = \mathbb{R}_{121} \otimes \mathbb{R}_{\mathbf{33}21} + \mathbb{R}_{121\mathbf{33}} \otimes \mathbb{R}_{21} + \mathbb{R}_{131\mathbf{44}2} \otimes \mathbb{R}_{1}$
- $\Delta_{\prec}(\mathbb{R}_{242\mathbf{55}31}) = \mathbb{R}_{121\mathbf{33}} \otimes \mathbb{R}_{21} + \mathbb{R}_{131\mathbf{44}2} \otimes \mathbb{R}_{1}$
- $\Delta_{\succ}(\mathbb{R}_{242\mathbf{55}31}) = \mathbb{R}_{121} \otimes \mathbb{R}_{\mathbf{33}21}$

Refinement of the Hopf definition

Refinement of associativity and coassociativity: 3 and 3 equations
Refinement of the Hopf relation: 4 equations

Totally primitive elements

Primitive elements Prim(A)

P is a primitive element $\iff \tilde{\Delta}(P) = 0$

Ex : $P = \mathbb{R}_{121\mathbf{3}} - \mathbb{R}_{2\mathbf{3}21}$ $\quad \tilde{\Delta}(P) = \mathbb{R}_{121} \otimes \mathbb{R}_{\mathbf{1}} - \mathbb{R}_{\mathbf{121}} \otimes \mathbb{R}_{1} = 0$

Totally primitive element TPrim(A)

P is a totally primitive element $\iff \Delta_{\prec}(P) = \Delta_{\succ}(P) = 0$

Ex : $\mathbb{R}_{12443} - \mathbb{R}_{21443} - \mathbb{R}_{23441} + \mathbb{R}_{32441}$

Theorem [Foissy]

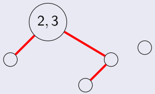
If A is a bidendriform bialgebra then A is freely generated by TPrim(A) as a dendriform algebra.

Corollary

WQSym is self-dual.

The basis $\mathbb{P}$ indexed by red biplan forests

Example of expansion from $\mathbb{P}$ to $\mathbb{R}$

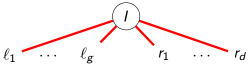
$\mathbb{P}$ 

$$= \mathbb{R}_{235541} - \mathbb{R}_{245531} - \mathbb{R}_{244531} - \mathbb{R}_{245431} - \mathbb{R}_{254431} + \mathbb{R}_{325541} - \mathbb{R}_{425531} - \mathbb{R}_{524431} + \mathbb{R}_{352541} - \mathbb{R}_{452531} + \mathbb{R}_{355241} - \mathbb{R}_{455231} + \mathbb{R}_{344521} + \mathbb{R}_{345421} + \mathbb{R}_{354421} + \mathbb{R}_{534421}$$

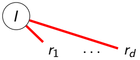
Bijection between packed words and red biplan forests. ?

$\mathbb{P}$  $:= \mathbb{R}_1,$

$$\mathbb{P}_{t_1,\dots,t_k} := (\dots(\mathbb{P}_{t_k} \prec \dots) \prec \mathbb{P}_{t_2}) \prec \mathbb{P}_{t_1},$$

$\mathbb{P}$ 

$$:= \langle \mathbb{P}_{\ell_1}, \mathbb{P}_{\ell_2}, \dots, \mathbb{P}_{\ell_k}; \Phi_l(\mathbb{P}_{r_1,\dots,r_d})\rangle$$

$\mathbb{P}$ 

$$:= \Phi_l(\mathbb{P}_{r_1,\dots,r_d}).$$

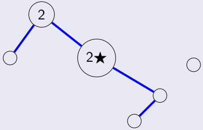
Theorem

$(\mathbb{P}_f)_{f\in\mathfrak{F}_n}$ for **WQSym**$_n$,
$(\mathbb{P}_t)_{t\in\mathfrak{T}_n}$ for Prim$_n$,
$(\mathbb{P}_t)_{t\in\mathfrak{P}_n}$ for TPrim$_n$.

The basis $\mathbb{O}$ indexed by blue biplan forests

Example of expansion from $\mathbb{O}$ to $\mathbb{Q}$

$\mathbb{O}$



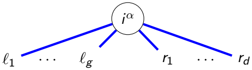
$$= \mathbb{Q}_{531442} + \mathbb{Q}_{521443} + \mathbb{Q}_{512443} - \mathbb{Q}_{534142} - \mathbb{Q}_{524143} - \mathbb{Q}_{514243} - \mathbb{Q}_{514432} - \mathbb{Q}_{524431} - \mathbb{Q}_{514423} + \mathbb{Q}_{541432} + \mathbb{Q}_{542431} + \mathbb{Q}_{541423}$$

Bijection between packed words and blue biplan forests. ❓

$$\mathbb{O}_{\circ} := \mathbb{Q}_1,$$

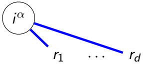
$$\mathbb{O}_{t_1,\dots,t_k} := (\dots(\mathbb{O}_{t_k} \prec \dots) \prec \mathbb{O}_{t_2}) \prec \mathbb{O}_{t_1},$$

$\mathbb{O}$



$$:= \langle \mathbb{O}_{l_1}, \mathbb{O}_{l_2}, \dots, \mathbb{O}_{l_k}; \Psi_i^\alpha(\mathbb{O}_{r_1,\dots,r_d}) \rangle$$

$\mathbb{O}$



$$:= \Psi_i^\alpha(\mathbb{O}_{r_1,\dots,r_d}).$$

Theorem

$(\mathbb{O}_f)_{f \in \mathfrak{F}^*{}_n}$ for $\mathbf{WQSym}_n^*$

$(\mathbb{O}_t)_{t \in \mathfrak{T}^*{}_n}$ for Prim_n^*,

$(\mathbb{O}_t)_{t \in \mathfrak{P}^*{}_n}$ for TPrim_n^*.

Bijection between red and blue biplan forests