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# QUASYSIMMETRIC INVARIANTS FOR FAMILIES OF POSETS

DORIANN ALBERTIN, JEAN-CHRISTOPHE AVAL, AND HUGO MLODECKI

ABSTRACT. We address the question whether certain classes of labeled posets can be distinguished by their P-partition enumerator. Recent progress have been made for *naturally labeled* posets [5, 10, 11]. We investigate the general question and prove that the P-partition enumerator does distinguish classes of labeled posets that we call: fair series-parallel posets, cypress trees, and centipedes.

#### 1. INTRODUCTION

1.1. Distinguishing combinatorial objects, a classical problem. Consider an invariant  $\Gamma$  on a class of graphs  $\mathcal{G}$ . It is a basic question, both classical and active in the recent years, to decide whether  $\Gamma$  is injective on (isomorphic classes of)  $\mathcal{G}$ . In this case we say that  $\Gamma$  distinguishes elements of  $\mathcal{G}$ .

Let us start by recalling the definition of the celebrated chromatic polynomial  $\chi$ . In the following definitions, all graphs are finite and simple.

**Definition 1.1.** A (proper) *coloration* of a graph G = (V, E) is a function

$$c: V \longrightarrow \mathbb{N}^*$$

such that for  $v, v' \in V$ ,  $(v, v') \in E \Rightarrow c(v) \neq c(v')$ .

It is well-known that the number of colorations of G over t colors turns out to be a polynomial in t, which we denote by  $\chi_G(t)$ , the *chromatic polynomial* of G. It is true that some informations can be derived from  $\chi_G$ , such as the number of vertices |V|, the number of edges |E|, or the number of connected components of G. But  $\chi$  is very far from being able to distinguish even simple classes of graphs, since for example  $\chi_T(t) = t(t-1)^{k-1}$  for any tree T with k vertices.

In order to distinguish classes of graphs, Stanley defined in 1995 a new chromatic invariant, stronger than the chromatic polynomial [18]. Let  $\mathbf{x} = x_1, x_2, \ldots$  denote an infinite commutative alphabet.

**Definition 1.2.** The chromatic symmetric function  $X_G$  of a graph G is defined as

$$X_G(\mathbf{x}) = \sum_c \mathbf{x}^c$$

where the sum ranges over the colorations c of G, and  $\mathbf{x}^c$  stands for  $\prod_{i \in \mathbb{N}^*} x_i^{|c^{-1}(i)|}$ .

In his original paper, Stanley observed that  $X_G$  does not distinguish graphs in general: he gave the example of two graphs on 5 vertices with equal chromatic symmetric functions. But he conjectured the following.

**Conjecture 1.3.** The chromatic symmetric function distinguishes trees: for any pair of trees  $T_1$  and  $T_2$ , if  $X_{T_1} = X_{T_2}$ , then  $T_1$  and  $T_2$  are isomorphic.

This conjecture has been intensively studied and proved for special subclasses of trees [15, 12, 7, 8]. Important variations have been proposed [17, 3, 13], but it is still open in general.

Let us now turn to an analogue of the previous question, in the setting of (labeled) posets instead of graphs.

1.2. The main question of this work: quasisymmetric invariant distinguishing labeled posets. Let P be a poset with n elements; we write |P| = n. Denote the order relation on P by  $\leq_P$ , to avoid the confusion with the usual order on the positive integers, which we shall note  $\leq$ . A *labeling* of P is a bijection  $\omega : P \to [n]$ . A *labeled poset*  $(P, \omega)$  is then a poset P with an associated labeling  $\omega$ .

**Definition 1.4.** For a labeled poset  $(P, \omega)$ , a  $(P, \omega)$ -partition is a map f from P to the positive integers satisfying the following two conditions:

- if  $a <_P b$ , then  $f(a) \le f(b)$ , i.e., f is order-preserving;
- if  $a <_P b$  and  $\omega(a) > \omega(b)$ , then f(a) < f(b).

In other words, a  $(P, \omega)$ -partition is an order-preserving map from P to the positive integers with certain strictness conditions determined by  $\omega$ . Examples of  $(P, \omega)$ -partitions f are given in Figure 1.

The meaning of the double edges in the figure follows from the following observation about Definition 1.4. For  $a, b \in P$ , we say that a is covered by b in P, denoted  $a \prec_P b$ , if  $a <_P b$  and there does not exist c in P such that  $a <_P c <_P b$ . Note that a definition equivalent to Definition 1.4 is obtained by replacing both appearances of the relation  $a <_P b$  with the relation  $a \prec_P b$ . In other words, we require that fbe order-preserving along the edges of the Hasse diagram of P, with f(a) < f(b)when the edge  $a \prec_P b$  satisfies  $\omega(a) > \omega(b)$ . With this in mind, we will consider those edges  $a \prec_P b$  with  $\omega(a) > \omega(b)$  as strict edges and we will represent them in Hasse diagrams by double lines. Similarly, edges  $a \prec_P b$  with  $\omega(a) < \omega(b)$  will be called *weak edges* and will be represented by single lines.

From the point-of-view of  $(P, \omega)$ -partitions, the labeling  $\omega$  only determines which edges are strict and which are weak. Therefore, we say that two labeled posets  $(P, \omega)$  and  $(Q, \omega')$  are *isomorphic* if P and Q are isomorphic as posets and they have equivalent sets of strict and weak edges according to a poset isomorphism.

**Definition 1.5.** An *edge-decorated poset* is a poset P such that each edge in the Hasse diagram of P is assigned to be either large or strict.

From now on, we will consider edge-decorated posets P instead of labeled posets  $(P, \omega)$ . Moreover, we shall use the notations:  $\overline{P}$  for the edge-decorated poset obtained by switching weak and strict edges in P, and  $P^{\uparrow}$  for the reverse (upside-down) poset with the same decoration of edges.

If all the edges of P are weak, as in Figure 1(b), P is said to be *weak*. This corresponds to order-preserving labelings  $\omega$ , and such P are called *naturally labeled* in some references.

**Definition 1.6.** Let P be an edge-decorated poset. The well-known P-partition enumerator is defined by

(1) 
$$K_P(\mathbf{x}) = \sum_f x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots$$

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FIGURE 1. Examples of  $(P, \omega)$ -partitions (the images are written in bold and blue next to the nodes)



FIGURE 2. Pairs of weak posets with equal *P*-partition enumerators

where the sum ranges over all *P*-partitions  $f: P \to \mathbb{P}$ .

A poset being a tree simply means its Hasse diagram is a tree. The following conjecture is presented in [1].

## Conjecture 1.7. The P-partition enumerator distinguishes weak trees.

If we try to relax the conditions in Conjecture 1.7, we obtain easily false statements. Even for weak posets. The first example of non-isomorphic weak posets with the same K was given in [16] and appears in Figure 2(a). A *bowtie* is the poset consisting of elements  $a_1, a_2, b_1, b_2$  with cover relations  $a_i < b_j$  for all i, j. Notice that each poset in Figure 2(a) has a bowtie as an induced subposet. Otherwise, we say the poset is *bowtie-free*. Weakening the tree hypothesis of Conjecture 1.7 to bowtie-free results in a false statement, with Figure 2(b) being the smallest counterexample.

A important result in this context is the following.

**Theorem 1.8** ([5], Theorem 1.3). The *P*-partition enumerator distinguishes weak rooted trees.

But Conjecture 1.7 is still open. Theorem 1.8 was generalised to weak seriesparallel posets in [10]. Up to very recently, the effort put on distinguishing posets through the P-partition enumerator has been focused on weak posets (we may add



FIGURE 3. Two simple edge-decorated posets with equal P-partition enumerator

[11, 21] to the references already mentionned). When we consider general edgedecorated posets (not necessarily weak), things are getting (way!) harder. Figure 3 shows how very simple edge-decorated posets may have the same K function. Of course, this example implies that there is no chance to extend Conjecture 1.7 to general edge-decorated trees.

Up to now, the first and only result for distinguishing edge-decorated posets was obtained in [1] and concerns a family named (rooted) fair trees.

It is the purpose of this article to present new results on this question. We prove here that the *P*-partition enumerator does distinguish families named: fair seriesparallel posets (Section 4), cypress trees (Section 5), and centipedes (Section 6).

#### 2. Definitions and useful tools

2.1. Quasisymmetric functions. We shall give here basic definitions and properties about quasisymmetric functions.

For our purposes, quasisymmetric functions are elements of  $\mathbb{Q}[[x_1, x_2, \ldots]]$  and we denote the ring of quasisymmetric functions by QSym. We will make use of both of the classical bases for QSym. If  $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$  is a composition of n, then we define the *monomial quasisymmetric function*  $M_{\alpha}$  by

$$M_{\alpha} = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

We recall that compositions  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$  of n are in bijection with subsets of [n-1], and let  $S(\alpha)$  denote the set  $\{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{k-1}\}$ . Thus we also denote  $M_{\alpha}$  by  $M_{S(\alpha),n}$ . Notice that these two notations are distinguished by the latter one including the subscript n; this subscript is helpful since  $S(\alpha)$  does not uniquely determine n.

The second classical basis is composed of the fundamental quasisymmetric functions  $F_{\alpha}$  defined by

(2) 
$$F_{\alpha} = F_{S(\alpha),n} = \sum_{S(\alpha) \subseteq T \subseteq [n-1]} M_{T,n}$$

The relevance of this latter basis to  $K_{(P,\omega)}$  is due to Theorem 2.1 below.

Recall that any permutation  $\pi \in S_n$  has an associated descent set  $des(\pi)$  given by  $\{i \in [n-1] : \pi(i) > \pi(i+1)\}$ . We will call the corresponding composition of nthe *descent composition* of the permutation  $\pi$ , denoted  $co(\pi)$ . As an example, if  $\pi = 243561$ , then  $des(\pi) = \{2, 5\}$  and  $co(\pi) = 231$ . Let  $\mathcal{L}(P, \omega)$  denote the set of all linear extensions of P, regarded as permutations of the  $\omega$ -labels of P. For example, for the labeled poset in Figure 1(a),  $\mathcal{L}(P, \omega) = \{1423, 1432, 4123, 4132\}$ .

**Theorem 2.1** ([4, 19, 20]). Let  $(P, \omega)$  be a labeled poset with |P| = n. Then

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\operatorname{des}(\pi),n} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\operatorname{co}(\pi)} \,.$$



FIGURE 4. The labeled poset of Example 2.2

**Example 2.2.** The labeled poset  $(P, \omega)$  of Figure 4 has  $\mathcal{L}(P, \omega) = \{1324, 1342\}$  and hence

$$\begin{split} K_{(P,\omega)} &= F_{\{2\},4} + F_{\{3\},4} \\ &= F_{22} + F_{31} \\ &= (M_{\{2\},4} + M_{\{1,2\},4} + M_{\{2,3\},4} + M_{\{1,2,3\},4}) \\ &+ (M_{\{3\},4} + M_{\{1,3\},4} + M_{\{2,3\},4} + M_{\{1,2,3\},4}) \\ &= M_{22} + M_{31} + M_{112} + 2M_{211} + M_{121} + 2M_{1111} \end{split}$$

**Remark 2.3.** It is easy to deduce the *P*-partition enumerator of the edge-decorated poset  $\overline{P}$  obtained by exchanging weak and strict edges in *P*. We may call  $\overline{P}$  the dual of *P*. Indeed, if  $K_P = \sum_{\alpha} c_{\alpha} F_{\alpha,n} = \sum_{\alpha} c_{\alpha} F_{S(\alpha),n}$  then  $K_{\overline{P}} = \sum_{\alpha} c_{\alpha} F_{\overline{S(\alpha)},n}$  where  $\overline{S}$  stands for the complementary set of the set *S* in [n-1].

For example, the dual edge-decorated poset of Figure 4 has its *P*-partition enumerator equal to  $F_{121} + F_{112}$ .

The following result appears in [6, 9, 14] and is crucial in our context, and more generally in these questions about the distinguishability of combinatorial families by the *P*-partition enumerator.

# **Theorem 2.4** ([6, 9]). QSym is a unique factorization domain.

2.2. Free quasisymmetric functions. We give a short survey of basic definitions and properties about free quasisymmetric functions that we will use.

The Hopf algebra **FQSym** of Malvenuto-Reutenauer, also called Hopf algebra of free quasi-symmetric functions ([2, 14]). The algebra **FQSym** is the vector space generated by the elements  $(\mathbb{F}_u)_{u\in\mathfrak{S}}$ , where  $\mathfrak{S}$  is the disjoint union of the symmetric groups  $S_n$   $(n \in \mathbb{N})$ . Its product and its coproduct are given in the following way: for all  $u \in S_n$ ,  $v \in S_m$ , by putting  $u = (u_1 \dots u_n)$ ,

$$\Delta(\mathbb{F}_u) = \sum_{i=0}^n \mathbb{F}_{\mathrm{st}(u_1\dots u_i)} \otimes \mathbb{F}_{\mathrm{st}(u_{i+1}\dots u_n)},$$
$$\mathbb{F}_u \cdot \mathbb{F}_v = \sum_{w \in u} \Delta_v \mathbb{F}_w,$$

where  $u \Delta v$  is the shifted shuffle of u and v, and st is the standardisation operator. Its unit is  $1 = \mathbb{F}_{\emptyset}$ , where  $\emptyset$  is the unique element of  $S_0$ . Moreover, **FQSym** is a  $\mathbb{N}$ -graded Hopf algebra, by putting  $|\mathbb{F}_u| = n$  if  $u \in S_n$ .

# Example 2.5.

$$\mathbb{F}_{(1\,2)}\mathbb{F}_{(1\,2\,3)} = \mathbb{F}_{(1\,2\,3\,4\,5)} + \mathbb{F}_{(1\,3\,2\,4\,5)} + \mathbb{F}_{(1\,3\,4\,2\,5)} + \mathbb{F}_{(1\,3\,4\,5\,2)} + \mathbb{F}_{(3\,1\,2\,4\,5)} \\ + \mathbb{F}_{(3\,1\,4\,2\,5)} + \mathbb{F}_{(3\,1\,4\,5\,2)} + \mathbb{F}_{(3\,4\,1\,2\,5)} + \mathbb{F}_{(3\,4\,1\,5\,2)} + \mathbb{F}_{(3\,4\,5\,1\,2)},$$

$$\Delta \left( \mathbb{F}_{(1\,2\,5\,4\,3)} \right) = 1 \otimes \mathbb{F}_{(1\,2\,5\,4\,3)} + \mathbb{F}_{(1)} \otimes \mathbb{F}_{(1\,4\,3\,2)} + \mathbb{F}_{(1\,2)} \otimes \mathbb{F}_{(3\,2\,1)} \\ + \mathbb{F}_{(1\,2\,3)} \otimes \mathbb{F}_{(2\,1)} + \mathbb{F}_{(1\,2\,4\,3)} \otimes \mathbb{F}_{(1)} + \mathbb{F}_{(1\,2\,5\,4\,3)} \otimes 1.$$

# 3. Statistics determined by the P-partition enumerator

Much is known in the case of naturally labeled posets (see a list of properties in [1]), by far less in the general case. In this section, we recall the theory of jumps initiated in [16], slightly extending a result of [10], and we give a necessary condition for two labeled posets to have the same P-partition enumerator in the special case of posets with exactly one minimal element.

3.1. **Jumps.** The notion of *jump* was first considered in [16]. We recall here the main definitions and results.

**Definition 3.1.** Let b be an element of a labeled poset P. Let us consider the number of strict edges on a saturated chains from b down to a minimal element of P. The *jump* of b is defined as the maximal such number. In a similar way the *up-jump* of b is obtained by considering saturated chains from b up to a maximal element of P.

We introduce the statistics:  $J_P^{\downarrow}(i)$  denotes the number of elements of jump equal to i in P,  $J_P^{\uparrow}(j)$  the number of elements of up-jump equal to j in P, and  $J_P(i,j)$ the number of elements of jump equal to i and up-jump equal to j in P.

The first result was obtained by McNarama and Ward [16] (Proposition 4.2) who proved that for any labeled poset P, the value of  $J_P^{\downarrow}(i)$  is determined for any i by K. In the naturally labeled case, Liu and Weselcouch proved ([10], Lemma 3.9) that for any i and j, the value of  $J_P(i, j)$  is determined by K. We extend this to the general case. Although it is quite similar to the aformentioned result, we consider it useful to state and prove this result, since it is a powerful tool to distinguish labeled posets.

**Proposition 3.2.** Let P and Q be edge-decorated posets. We have, for any i and j:

(3) 
$$K_P = K_Q \Longrightarrow J_P(i,j) = J_Q(i,j).$$

*Proof.* We shall use Corollary 5.3 in [16] which asserts that

$$K_P = K_Q \Longrightarrow K_{P_{[i]}} = K_{Q_{[i]}}$$

where  $P_{[i]}$  denotes the restriction of P to elements of jump at least i. Thus we have also that  $K_{P_{[(i,j)]}} = K_{Q_{[(i,j)]}}$  where  $P_{[(i,j)]}$  denotes the restriction of P to elements of both jump at least i, and up-jump at least j. By a simple degree consideration, we get:  $|P_{[(i,j)]}| = |Q_{[(i,j)]}|$ . We conclude by observing that  $J_P(i,j) = |P_{[(i,j)]}| - |P_{[(i,j+1)]}| - |P_{[(i+1,j+1)]}|$ .

3.2. **Posets with one minimal element.** Now, the following lemma is very useful when studying the *P*-partition enumerator of edge-decorated posets.

**Lemma 3.3.** Let P be an edge-decorated poset. If P has one minimal element (that we denote by  $v_0$ ), the P-partition enumerator of  $P \setminus v_0$  may be computed from  $K_P$ .

Proof. We use Theorem 2.1 and consider the linear extensions  $\sigma \in \mathcal{L}(P)$ . Let us denote by  $a_1, \ldots, a_k$  the elements of P that cover  $v_0$  with strict edges, and  $b_1, \ldots, b_l$ the elements of P that cover  $v_0$  with weak edges. Such a  $\sigma$  may be of two types: it always start with  $v_0$ , followed by either an  $a_i$  or a  $b_i$ . In the first case  $\sigma$  has an initial descent, thus  $\mathbf{F}_{\operatorname{des}(\sigma),n}$  has a first part at least 2. In the second case  $\sigma$  has an initial ascent, thus  $\mathbf{F}_{\operatorname{des}(\sigma),n}$  has a first part equal to 1. Thus we can decompose the P-partition enumerator K in two parts:  $K = K_1 + K_2$  with  $K_1$  consisting in the  $\mathbf{F}_{\alpha}$ with  $\alpha_1 = 1$  and  $K_2$  consisting in the  $\mathbf{F}_{\alpha}$  with  $\alpha_1 \geq 2$ . Since the linear extensions of  $P \setminus v_0$  are just those of P without the initial  $v_0$ , the P-partition enumerator of  $P \setminus v_0$  is  $K'_1 + K'_2$  where  $K'_1$  is deduced from  $K_1$  by removing the first part (equal to 1) in any  $\mathbf{F}_{\alpha} \in K_1$ , and  $K'_2$  is deduced from  $K_2$  by substracting 1 to the first part (greater or equal to 2) in any  $\mathbf{F}_{\alpha} \in K_2$ .

# 4. FAIR SERIES-PARALLEL POSETS

In this section, we introduce a subclass of edge-decorated posets: *fair series* parallel posets. We prove it is distinguished by the *P*-partition enumerator. This result stands in the direct continuity of the article [1]. Beforehand, we give some precise statements about quasisymmetric and free quasisymmetric functions.

**Definition 4.1.** A *fair series parallel poset* is an edge decorated poset recursively defined as either:

- A single element [1],
- The poset  $P \sqcup Q$  for any series-parallel posets P and Q,
- The poset  $P \uparrow Q$  for any series-parallel posets P and Q, obtained from  $P \sqcup Q$  by adding a weak edge (p, q) for all pairs of maximal element p of P and minimum element q of Q,
- The poset  $P \Uparrow Q$  for any series-parallel posets P and Q, obtained from  $P \sqcup Q$  by adding a strict edge (p,q) for all pairs of maximal element p of P and minimum element q of Q.

See Figure 5.

Fair series parallel posets are a natural generalization of fair trees of [1].

**Remark 4.2.** Fair series-parallel posets where all edges are weak correspond to classical series-parallel posets. They are exactly the N-free posets. It is already known that they are distinguished by strict P-partition enumerators, see [10]. It remains open to characterize fair series-parallel posets as minor free posets.

Without further ado, we state the expected result:

### **Theorem 4.3.** *P*-Partition enumerators distinguish fair series parallel posets.

We emphasize the fact that there are, to our knowledge, only two theorems about injectivity of P-partition enumerators over families of posets with both weak and strict edges: Theorem 4.4 of [1], and our theorem which generalizes the latter.



FIGURE 5. Examples of connected fair series-parallel posets. The second one is in particular a fair tree as of the definition of [1].

Following a recurrent pattern among proof of such injectivity theorems (see [5, 11]), we will heavily rely on the following technical irreducibility lemma:

**Lemma 4.4.** The *P*-partition enumerator of a connected fair series-parallel poset is irreducible in QSym.

*Proof.* Let P be a connected fair series-parallel poset. Without loss of generality, assume that  $P = Q \Uparrow R$  for some fair series-parallel posets Q and R (the other case being dual, thanks to Remark 2.3). Then all linear extensions of P admit the same descent q := |Q|. We will show that this *global descent*, along with the homogeneity of P-partition enumerators, implies irreducibility.

For contradiction, assume that  $K_P = fg$  for some non-trivial  $f, g \in QSym$ . Since  $K_P$  is homogeneous (say, of degree n), then so are f and g (say, of degrees  $n_1$  and  $n_2$  with  $n_1 + n_2 = n$ ). Let  $c_{\alpha}$ ,  $d_{\beta}$  and  $e_{\delta}$  be the coefficients of  $K_P$ , f and g, so that :

(4) 
$$\sum_{\alpha \models n} c_{\alpha} \boldsymbol{F}_{\alpha} = K_{P} = fg = \left(\sum_{\beta \models n_{1}} d_{\beta} \boldsymbol{F}_{\beta}\right) \left(\sum_{\gamma \models n_{2}} e_{\gamma} \boldsymbol{F}_{\gamma}\right).$$

Then, lifting everything up to **FQSym**:

(5) 
$$\sum_{\sigma \in \mathfrak{S}_n} c_{\operatorname{des}(\sigma),n} \mathbb{F}_{\sigma} = \left( \sum_{\nu \in \mathfrak{S}_{n_1}} d_{\operatorname{des}(\nu),n_1} \mathbb{F}_{\nu} \right) \left( \sum_{\tau \in \mathfrak{S}_{n_2}} e_{\operatorname{des}(\tau),n_2} \mathbb{F}_{\tau} \right),$$

(6) 
$$= \sum_{\sigma \in \mathfrak{S}_n} \left( \sum_{\sigma \in \nu \Delta \tau} d_{\operatorname{des}(\nu), n_1} e_{\operatorname{des}(\tau), n_2} \right) \mathbb{F}_{\sigma},$$

the second sum in the last right hand side term being over  $\nu$  and  $\tau$ . Observe that given any permutation  $\sigma \in \mathfrak{S}_n$ , there is only one pair  $(\nu, \tau) \in \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$  such that  $\sigma \in \nu \Delta \tau$ . Indeed,  $\nu = \sigma^{[1,n_1]}$  is the permutation describing the  $n_1$  smallest values of  $\sigma$ , while  $\tau = \sigma^{[n_1,n_1]}$  is the permutation describing the  $n_2$  biggest values of  $\sigma$ . Hence we get :

(7) 
$$\sum_{\sigma \in \mathfrak{S}_n} c_{\operatorname{des}(\sigma),n} \mathbb{F}_{\sigma} = \sum_{\sigma \in \mathfrak{S}_n} d_{\operatorname{des}(\sigma^{[1,n_1]},n_1)} e_{\operatorname{des}(\sigma^{]n_1,n]},n_2)} \mathbb{F}_{\sigma}.$$

Since  $K_P$  has a global descent q, it follows that

$$\forall \sigma \in \mathfrak{S}_n, q \notin \operatorname{des}(\sigma) \Rightarrow 0 = c_{\operatorname{des}(\sigma),n} = d_{\operatorname{des}(\sigma^{[1,n_1]}),n_1} e_{\operatorname{des}(\sigma^{[n_1,n_1]},n_2)}.$$

For all  $\beta \in \mathfrak{S}_{n_1}$  and  $\gamma \in \mathfrak{S}_{n_2}$  we let  $\sigma_{\beta,\gamma}$  be a permutation such that :

- $q \notin \operatorname{des}(\sigma_{\beta,\gamma}),$
- $\operatorname{des}(\sigma_{\beta,\gamma}^{[1,n_1]}) = \beta$ ,  $\operatorname{des}(\sigma_{\beta,\gamma}^{[n_1,n_1]}) = \gamma$ .

It is not hard to check that such a permutation always exists. Plugging  $\sigma_{\beta,\gamma}$  into Equation (7) yields:

$$\forall \beta \in \mathfrak{S}_{n_1}, \gamma \in \mathfrak{S}_{n_2}, d_\beta \cdot e_\gamma = 0,$$

and it follows that either f = 0 or g = 0, which is absurd.

*Proof of Theorem 4.3.* We proceed by induction on the size of the posets.

Let P and P' be fair series-parallel posets such that  $K_P = K_{P'}$ . We distinguish according to the shape of P. If  $P = \bigsqcup_i P_i$  with every  $P_i$  a connected fair seriesparallel poset, then  $K_P = K_{\bigcup_i P_i} = \prod_i K_{P_i}$ . On the other hand,  $K_{P'} = K_{\bigcup_i P'_i} =$  $\prod_i K_{P'_i}$  where  $P' = \bigsqcup_i P'_i$ . Unique factorization in QSym and Lemma 4.4 conclude by induction.

If  $P = Q \Uparrow R$ , then as before, all linear extensions of P admit |Q| as a descent, and same goes for P' since  $K_P = K_{P'}$ . We claim that  $P' = Q' \uparrow R'$  for some fair series-parallel posets Q' and R' with |Q| = |Q'| =: q. Showing this amounts to showing that all linear extensions of P' have the same set of q first entries. Assume by contradiction that  $\sigma$  and  $\nu$  are two linear extensions of P' such that there exist  $i \in \sigma_{[1,q]} \setminus \nu_{[1,q]}$  and  $j \in \nu_{[1,q]} \setminus \sigma_{[1,q]}$ , and take such *i* (resp. *j*) maximizing (resp. minimizing) the position of i (resp. j) in  $\sigma$ . If the position of i in  $\sigma$  is less than q, denote by k the integer just right of i in  $\sigma$ . Since i is of maximal index in  $\sigma$ , we have  $k \in \nu_{[1,q]}$ , so k appears left of i in  $\nu$  and right of i in  $\sigma$ , and i and k are incomparable in P'. As a result, exchanging i and k in  $\sigma$  yields another linear extension of P'. By induction, we may move i all the way to position q in  $\sigma$ , and the resulting permutation would still be a linear extension of P'. We proceed similarly to move j to the position q+1 in  $\sigma$ . We've just built a linear extension  $\ell$  of P' with  $\ell_q = i$  and  $\ell_{q+1} = j$ . Since *i* and *j* are incomparable in P' (they appear in different orders in  $\sigma$  and  $\nu$ ),  $\ell \cdot (i, j)$  is also a linear extension of P'. One of these two extensions admits q as an ascent, which is absurd.

Since  $K_P = K_Q \Uparrow K_R$ , we can compute  $K_Q$  (resp.  $K_R$ ) from  $K_P$  by restraining all compositions in the support of  $K_P$  (in the monomial basis) to it's first (resp. last) blocks summing up to |Q| (resp. |R|). Same goes for  $K_{P'} = K_{Q'} \uparrow K_{R'}$ , and we deduce that  $K_Q = K_{Q'}$  and  $K_R = K_{R'}$ , and conclude by induction.

The case  $P = Q \uparrow R$  is treated similarly.

#### 5. Cypress trees

**Definition 5.1.** A cypress tree is an edge-decorated poset consisting in a rooted chain with either weak or strict edges (the trunk), on which are glued leaves by weak edges.

Figure 6 shows two examples of cypress trees.

**Proposition 5.2.** Cypress trees are distinguished by the P-partition enumerator.

The key point is the following lemma.

**Lemma 5.3.** Let C be a cypress tree whose P-partition enumerator is K. It is possible to get from K the number of leaves at the root in C.



FIGURE 6. Cypress trees

*Proof.* Let us denote by a the number of leaves at the root in C.

We first deal with the case where all the edges of C are weak, which is easily tested from K, because this is equivalent to  $J_C^{\downarrow}(1) = 0$ . In this case we have:  $a = J_C^{\downarrow}(1) - 1$ .

Thus we may suppose now that C has at least one strict edge. Let us denote by m > 0 the number of such strict edges. We know that m may be derived from K, it is simply the maximal value of the jump statistic.

Next let us introduce the number t of vertices of the trunk (excluding the root), whose jump is zero, which coincides with the vertices of the trunk connected to the root by only weak edges. We readily observe that t is exactly the number of vertices in C with both jump equal to zero and up-jump different from zero (this positive value being nothing but m).

If the first edge of the trunk is strict (ie. t = 0), then we have:  $a = J_C^{\downarrow}(0)$ . And if the first two edges of the trunk are weak (ie. t > 1), then we have:  $a = J_C^{\downarrow}(1) - 1$ .

Thus we are left with the case where the first edge of the trunk of C is weak, and the second strict (ie. t = 1). We need also to define the following condition: we say that the trunk of C is a 1-trunk if it has a weak edge at the root and then only strict edges (remember we are in the case t = 1). We may use K to test whether the trunk of C is a 1-trunk, by testing that  $J_C(i, j)$  returns 1 for  $(i, j) \in \{(m, 0), (m - 1, 1), \ldots, (1, m - 1)\}$ .

We now introduce  $l = J_{\overline{C}}(1,0)$ . Then l = a unless the trunk of C is a 1-trunk, in which case l = a + X, where X is the number of vertices (of the trunk) without any leaf as a descendant.

So now, if the trunk C is a 1-trunk, we consider the maximal value k such that  $J_C^{\downarrow}(m) = 1, \ J_C^{\downarrow}(m-1) = 1, \ \ldots, \ J_C^{\downarrow}(m-k+1) = 1$ . Let us denote by  $v_1$  and  $v_2$  the first and second vertex of the trunk of C. We distinguish two cases:

- if k < m (ie.  $v_2$  does not contribute to k), then X = k whence a = l k;
- if k = m (ie.  $v_2$  does contribute to k), we have to decide whether  $v_1$  has a positive number of leaves (in which case X = k) or not (in which case X = k + 1). We are done by observing that the number of leaves of  $v_1$  is just  $b = J_{\overline{C}}^{\downarrow}(2)$ .



FIGURE 7. Three A-centipedes with  $A = [, \|, \|$  and the trunk depicted in orange.

*Proof of Proposition 5.2.* We will prove that we are able to reconstruct a cypress tree from its *P*-partition enumerator. The proof is based on an induction on the number of edges, the case of cypress with one edge being trivial.

So, let us consider a cypress tree C with n > 1 edges, whose P-partition enumerator is K. We know by Lemma 5.3 that we are able to get from K the number of leaves at the root in C.

We may apply Lemma 3.3 to compute the *P*-partition enumerator K' of the edgedecorated poset C' obtained by erasing the root of C. Then we have:  $K' = \mathbf{F}_1^a \times L$ where L is the *P*-partition enumerator of the cypress tree D consisting in erasing the root and its leaves in C. We may now use the induction to derive D from L.  $\Box$ 

#### 6. Centipedes

**Definition 6.1.** Let A be a word in the two letters alphabet  $\{|, \|\}$ . A A-centipede is an edge-decorated poset consisting in a rooted chain whose edges are strict or weak according to A (its *body*), on which is glued any number of up-going weak edges and down-going strict edges (its *legs*).

Figure 7 shows examples of centipedes.

**Proposition 6.2.** For any fixed  $A \in \{|, \|\}^*$ , A-centipedes are distinguished by the *P*-partition enumerator.

*Proof.* We prove this statement by exhibiting the reverse bijection. Let  $A \in \{|, \|\}^n$  be a word with  $n_1$  letters | and  $n_2$  letters || and C be an A-centipede. For  $i \in [n+1]$ , call  $a_i$  (resp.  $b_i$ ) the number of weak (resp. strict) legs attached to the i<sup>th</sup> node of the body of C (starting at the bottom element).

Let k be number of |'s at the beginning of A, and  $\ell \leq k+1$  be the maximum integer such that  $b_1 = b_2 = \ldots = b_\ell$ . We show that we recover  $\ell$  from the P-partition enumerator  $K_C$ .

If k = 0, *i.e.*  $A_0 = \|$ , then  $b_1 = J_C^{\uparrow}(n_2+1)$  and we know whether  $\ell = 0$  or  $\ell = 1$ . If k > 0,  $J_C^{\uparrow}(n_2+1) = b_1+b_2+\ldots+b_k$ . If  $b_1 \neq 0$ , then  $J_C^{\downarrow}(0) = \sum_{i=1}^{n+1} b_i$ , and otherwise  $J_C^{\downarrow}(0) > \sum_{i=1}^{n+1} b_i$ . Since  $A_1 = |, J_C^{\downarrow}(0) = \sum_{i=1}^{n+1} b_i + 1$ , and we know whether  $b_1 = 0$ . Proceeding by induction, suppose  $b_1 = b_2 = \ldots = b_{k'}$  for some k' < k. Then in

the same way,  $J_C^{\downarrow}(0)$  equals (resp. is greater than)  $\sum_{i=1}^{n+1} b_i + \sum_{i=1}^{k'} (a_i + 1)$ . Since  $J_C^{\downarrow}(i) = a_i + 1$  for  $1 \le i \le k$ , we know whether  $b_{k'+1}$  is null. In the end, we read the value of  $\ell$  out of  $K_C$  through jumps.

Let's prove we can recover all  $a_i$ 's, starting with  $a_{n+1}$ . If  $A_n = |$ , then  $a_{n+1} = J_{\overline{C}}^{\downarrow}(n_1 + 1)$ , and otherwise  $a_{n+1} = J_{\overline{C}}^{\downarrow}(n_2 + \delta_{\ell < k+1}) - 1$ . By induction, assume we know  $a_{n+1}$  down to  $a_{i+1}$ , and denote by  $m_1$  (resp.  $m_2$ ) the number of | (resp. ||) in A between positions 1 and i - 1 (included). If  $A_{i-1} = |$ , then  $J_{\overline{C}}(m_1 + 1)$  is the sum of  $a_i$ , some  $a_j$  for j > i and a constant. In the same way, if  $A_{i-1} = ||$ , then  $J_{\overline{C}}(m_2 + \delta_{\ell < k+1})$  is the sum of  $a_i$ , some  $a_j$  for j > i and a constant. These expressions only depend on  $\ell$ , k, previous  $a_j$ 's and the form of A, and hence can be read in the P-partition enumerator.

By defining dually  $\overline{k}$  as the number of  $\|$ 's at the end of A and  $\overline{\ell}$  as the number of nodes on top of the body of C with no weak leg, and performing the same kind of computations, one recovers the values of the  $b_i$ 's. One can alternatively invoke a duality argument by considering the poset  $\overline{C}^{\uparrow}$  whose P-partition enumerator contains the same information as  $K_C$ .

**Remark 6.3.** It is not very difficult to adapt the proof to show that the *P*-partition enumerator distinguishes an *A*-centipede *C* from an *A'* centipede *C'* when *A* is a prefix of *A'* and both centipedes *C* and *C'* have a strict leg attached to their bottom element and a weak edge attached to their top element. We have no evidence that this leg condition is necessary, but it allows the proof to work easily.

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